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An Investigation of Secondary School Algebra Teachers' Mathematical Knowledge for Teaching Algebraic Equation Solving

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An Investigation of Secondary School Algebra Teachers' Mathematical Knowledge for Teaching Algebraic Equation Solving

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Dedication

To my grandparents

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An Investigation of Secondary School Algebra Teachers' Mathematical Knowledge for Teaching Algebraic Equation Solving

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This study characterizes the mathematical knowledge upon which secondary school algebra teachers draw when pondering problem situations that could arise in the teaching and learning of solving algebraic equations, as well as examines the potential connections between teachers' knowledge and their academic backgrounds and teaching experiences. Seventy-two middle school and high school algebra teachers in Texas participated in the study by completing an academic background questionnaire and a written-response assessment instrument. Eight participants were then invited for follow-up semi-structured interviews.

The results revealed three topic areas in equation solving in which teachers' mathematical subject matter understanding should be strengthened: (a) the balancing method, (b) the concept of equivalent equations, and (c) the properties of linear equations in their general forms. The participants provided a wide range of instances of student

misconceptions and difficulties in learning how to solve linear and quadratic equations, as well as a variety of strategies for helping students to improve their understanding. Teachers' subject matter knowledge played a central or prerequisite role in their reasoning and decision-making in specific contexts.

When the problem contexts became broader or more general, teachers drew from across the three basic domains of mathematical knowledge for teaching (knowledge of the mathematical subject matter, knowledge of learners' conceptions, and knowledge of didactic representations) and showed individual preferences. Overall, teachers tended to rely more heavily upon their knowledge of students' specific or general learning characteristics.

Statistical analyses suggest that teachers who majored in mathematics and who had the most experience in teaching first-year or more advanced algebra courses performed significantly higher on the assessment than their counterparts, and there is a linear relationship between teachers' performance and the number of advanced mathematics course they have taken. Neither course-taking in mathematics education nor number of years of algebra teaching made a significant difference in their performance. Results are either unclear or inconsistent about the role of teachers' (a) use of algebra textbooks, (b) prior experience with a method or a manipulative, and (c) participation in professional development activities. Teachers also rated (a) collaborating with and learning from colleagues and (b) dealing with student conceptions and questions as highly influential on their professional knowledge growth.

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Chapter 1 Introduction

Purposes of the Study

The present dissertation study deals with a current priority research area in mathematics education, teachers' mathematical knowledge for teaching, and is particularly focused on one of the fundamental topics of school algebra, solving linear and quadratic equations. The study has two major purposes:

1. To characterize the nature and extent of secondary school algebra teachers' use of mathematical knowledge for teaching algebraic equation solving.

2. To examine the potential connections between teachers' academic backgrounds and teaching experiences and their mathematical knowledge for teaching.

This is a descriptive and exploratory study.

Background

My research topic and objectives have been shaped mainly by four related issues in contemporary mathematics education research and practice:

1. There has been pressing demand for conceptualizing and measuring the mathematical knowledge that teachers may or should draw upon in teaching, especially at the secondary school level.

2. Such demand is even stronger for secondary school algebraic topics because of the increased accessibility to and requirements for algebra courses, as well as the diverse algebra curriculum standards and textbooks.

3. Equations and equation solving are fundamental topics in school algebra, but the literature on mathematics teachers' understanding of these topics is relatively scarce.

4. Equation solving involves important notions, such as mathematical procedures, algorithms, routines, skills, conceptual understanding, and procedure knowledge. Scrutinizing mathematics teachers' conceptions of these notions could provide crucial information for better understanding their teaching practices and influences on student learning.

The above issues are discussed in the following sections. Relevant literature will be summarized and analyzed in Chapter 2.

Mathematical Knowledge for Teaching as a Focal Area for Research

In a report on improving the mathematical proficiency of all students, the RAND Mathematics Study Panel (2003) identified three focal research areas for building a strategic research and development program in mathematics education: (a) *the teaching and learning of algebra for mathematical proficiency*, (b) *the teaching and learning of mathematical practices*, and (c) *the knowledge of mathematics needed for teaching*. My research topic falls within the intersection of the first and the third areas.

Although widespread agreement exists that effective mathematics teaching depends on teachers' knowledge of the mathematics content, the nature of such knowledge is still underspecified (RAND Mathematics Study Panel [RAND], 2003).

Further, in order to produce valid and consistent research findings about the relationships between teacher knowledge and student performance, indicators of teacher knowledge need to be sufficiently sensitive to measure the kind of knowledge that is most likely to impact student learning (Monk, 1994; Rowan, Chiang & Miller, 1997; Rowan, Correnti & Miller, 2002; Wilson, Floden & Ferrini-Mundy, 2001).

There have been increased efforts by mathematics education researchers in characterizing the kinds of mathematical knowledge that teachers may draw upon, bring to bear, or need for effective teaching (Ball, 2000; Ball & Bass, 2000, 2003; Ferrini-Mundy & Findell, 2001; Kennedy, 1997; Ma, 1999; Wilson, Shulman & Richert, 1987). Several groups of scholars have formulated perspectives on teachers' content knowledge for teaching in general (Shulman, 1986a, 1986b, 1987), knowledge of mathematics for teaching in general (Ball & Bass, 2005), and mathematical knowledge for teaching in a particular content area, such as school algebra (Artigue, Assule, Grugeon, & Lenfant, 2001; Jacobs, Borko, & Clark, 2006; Ferrini-Mundy, Burrill, Floden & Sandow, 2003) or geometry (Wing & Driscoll, 2006). Some of these perspectives will be reviewed in details in Chapter 2. Instruments for assessing knowledge of mathematics for teaching have been developed and disseminated by different institutions (Ball, Bass, Hill, & Schilling, 2005; Bush, 2005) and are being piloted and validated by research groups (Ferrini-Mundy, 2006; Floden & McCrory, 2007). At the present time, robust assessment instruments and results have been produced mostly for elementary and middle school level mathematics concepts, such as numbers and operations, proportional reasoning, and early algebra. More theoretical discussions and applicable instruments are desirable for secondary school level mathematical topics.

Teachers' mathematical knowledge for teaching and its use in teaching is highly relevant to the context for teaching and learning, which includes variables such as the topics and subjects being taught, state and local curriculum policies, and student learning characteristics. This assumption leads to the discussions in the next two sections.

Opportunities and Challenges in Teaching School Algebra

The teaching and learning of school algebra in the United States is a great endeavor with unique challenges. Traditionally, algebra was taught to (a) prepare a small group of students for college studies and (b) filter their educational opportunities. In the past two decades, the major role of algebra has expanded to providing tools for students to (a) describe patterns, changes, and relations in daily life; (b) formulate and resolve real-world problems; and (c) prepare them to become more knowledgeable participants in the workforce (U.S. Department of Education, 1997). Some educators and education reformers even referred to algebra as a new civil right and have devoted themselves to nationwide movements (Moses, 1995; Moses & Cobb, 2001). *Algebra for All* has become a goal of professional organizations (Achieve, Inc., 2004; Edwards, 1990; National Council of Teachers of Mathematics [NCTM], 1989, 2000) as well as state and local educational agencies. In 2004, 17 of the states require students to take at least one credit in algebra for high school graduation (Council of Chief State School Officers [CCSSO], 2005a), while 95% of high school students have actually taken first-year algebra or equivalent courses before graduation, and 72% of the students have taken second-year algebra or equivalent courses prior to graduation (CCSSO, 2005b; National Center for Education Statistics [NCES], 2004).

On the other hand, school algebra in the US "has become a deeply contested area for curriculum" (RAND, 2003, p.18). Central to the debates have been questions like "What is algebra?" and "What algebra topics should students learn? When? And in what ways?" A historical and global context for these questions is the coexistence of multiple perspectives on (or conceptions of) the nature and focuses of school algebra, such as generalized arithmetic, the study of structure, modeling, problem solving, and the study of relations and functions (Bednarz, Kieran, & Lee, 1996; Chazan, 2000; Stacey, Chick, & Kendal, 2004; Sutherland, Rojano, Bell, & Lins, 2001; Usiskin, 1988). These multiple perspectives have not only shaped some major themes for research on the learning and teaching of algebra (Wagner & Kieran, 1989) but also have been reflected by the diversity in the intended algebra curricula (including standards and instructional materials) across state and local settings in the American education system. A joint report by the National Council of Teachers of Mathematics and the Association of State Supervisors of Mathematics (Lott & Nishimura, 2004) compares the level of agreement on the coverage of mathematics topics across 16 state curriculum standards. Of 68 Algebra 1 topics appearing in these standards, only nine of the topics are agreed upon by 50% or more of the state standards, and 35 of the topics are mentioned in less than 25% of the standards. For 6th, 7th, and 8th grade algebra, the number of topics that are agreed upon by no less than 50% of the states are 0, 2, and 3, respectively. These results are consistent with the findings from a more recent study (Reys, 2006) on K-8 mathematics standards in 42 states. According to the report, although there is a core agreement among at least half of the states about which topics to include in K-8 algebra, “there appears to be little overall agreement across documents in the algebra expectations for a particular grade level” (p.8). Some of the studies in which my colleague and I have been involved (Li, 2005; Li & Zhao 2005) also revealed considerable structural differences between algebra standards in California and Texas; differences also appeared among 16 Algebra 1 textbooks used in American schools, in terms of the emphasis on and sequencing of two basic algebra themes: equations and functions.

The above discussions on the nature and roles of algebra as well as the reality of school algebra curriculum are crucial to the RAND Mathematics Study Panel’s decision to select algebra as one of the three priority subject areas for research (RAND, 2003). There exists the need to understand better the extent to which algebra teachers’ knowledge for teaching is shaped by various policy and curriculum variables, and how such knowledge may, in turn, shape the enacted algebra curricula and student learning (Lloyd, 1999; Lloyd & Wilson, 1998; Wilson & Lloyd, 2000).

A specific focus of my interest in studying knowledge for teaching algebra is about equations and equation solving. These topics have been central to algebra ever since its earliest development as a mathematical discipline in history (Bashmakova & Smirnova, 2000; Kline, 1972; van der Waerden, 1985). They also have been key components of school algebra curricula in the US during the entire 20th century (Donoghue, 2003). In the current school algebra curricula, equation solving typically builds upon fundamental algebraic concepts, such as generalized arithmetic, variables, and expressions, and it involves essential algebraic skills, such as identifying relationships between variables and performing symbolic manipulations and transformations. Building and solving equations are key steps in solving a variety of problems algebraically and are closely tied to the study of other core algebraic topics, such as systems of equations, inequalities, and functions. Both of the aforementioned reports on state mathematics standards (Lott & Nishimura, 2004; Reys, 2006) show that solving linear and quadratic equations is among the few algebraic topics on which at least half of the state standards studied agree. Linear and quadratic equations are also among the topics that are commonly covered by some widely used Algebra 1 textbooks. Details will be discussed in Chapter 2, Literature Review.

Teachers' Knowledge of Equation Solving Deserves More Attention

The need for studying algebra teachers' knowledge for teaching equation solving is further reinforced by two related but contrasting phenomena in mathematics education research: the existence of a relatively rich literature on how students understand equations and equation solving versus the scarce amount of studies on mathematics teachers' understanding of the same topic.

In the past three decades, various research studies have focused on students' understanding of different aspects of algebraic equations and equation solving, for instance, their conceptions of equations (Herscovics & Kieran, 1980; Kieran, 1979, 1982, 1984, 1988, 1989, 1997; Pirie & Martin, 1997), equivalence of equations (Steinberg, Sleeman, & Ktorza, 1990), and the development of reasoning in equation solving (Adi, 1978; Kieran, 1988; Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996).

Findings from these studies partially demonstrate the complexity of learning and teaching equations and equation solving, and they could be valuable resources for algebra teachers to enrich their knowledge for teaching algebra.

A few studies were conducted on preservice and inservice mathematics teachers' understanding of equations and equation solving, and yielded some interesting findings. For instance, preservice teachers may be able to recall correctly routines for solving equations but lack flexibility in applying the routines in the actual solving processes or other problem solving situations (Eisenburg & Dreyfus, 1988). Teachers, however, may emphasize the procedural aspects of the equation solving processes over the conceptual underpinnings (Attorps, 2003). Teaching a function-based, technology-intensive algebra curriculum seems to help a teacher to develop new perspectives on equations and equation solving but also reveals weak areas in one's conceptual understanding (Chazan, Larriva & Sandow, 1999; Yerushalmy, Leikin & Chazan, 2004).

Although intriguing, the above studies on teacher knowledge are mostly isolated efforts, with either single cases or small samples (less than 30) of participants involved; therefore, they are not sufficient in portraying the entire landscape of mathematics teachers' understanding of equations and equation solving. Doerr (2004) pointed out that much of the research on mathematics teachers' content knowledge of school algebra has focused on the concept of function, with more recent work addressing the concepts of slopes, variables, and expressions. Teachers' understanding of equations and equation solving is not among researchers' major focuses.

Besides filling the gaps in research, more systematic studies on this topic would be beneficial in the following ways: (a) Their design could directly utilize the existing findings on how students understand equations and equation solving, and their results could, in turn, relate, verify, or extend the findings on student learning; (b) They could produce detailed information and findings about the ways in which secondary school mathematics teachers apply their mathematical knowledge in interpreting and implementing the intended algebra curricula, and interacting with their students during instructional activities; and (c) such findings could also become resources for designing more effective mathematics teacher preparation curricula and professional development activities. My proposed study will address partially these three aspects, with a particular focus on equation solving.

Equation Solving as a Basic Mathematical Procedure and Algebraic Skill

Investigating teachers' conceptions of equation solving is also important because it is one of the basic mathematical procedures to be studied at the secondary school level and it involves a variety of basic algebraic and arithmetic skills, such as simplifying, factoring, or evaluating expressions, and applying the quadratic formula. Discrepancies have existed in the second half of the 20th century among the experiences and beliefs of mathematics educators, education researchers, mathematicians, education administrators, and policy makers regarding the nature and role of mathematical procedures, algorithms, knowledge of procedures, basic skills, procedural knowledge, and the relationships between procedural knowledge and conceptual understanding. The inception of the "New Math" in the 1950s was considered "the collision between skills instruction and understanding" (Bosse, 1995, p. 180). Despite the important successes in the New Math period, some of the school mathematics curricula were excessively formal, with little attention to basic skills or to applications of mathematics (Askey, 2001; Klein, 2003). The need for communication and consensus had never been stronger when the role of basic skills and their relationship with conceptual understanding became one of the central issues for debate during the "Math War" era of the late 1990s (Battista, 1999; Becker & Jacob, 2000; Wu, 1996, 1997, 1999).

In the last two decades, mathematics educators and education researchers have engaged in discussions about the relationships between procedural knowledge and conceptual understanding (Hiebert, 1986) and the role of algorithms in the teaching and learning of school mathematics (Morrow & Kenney, 1998). Specific to school algebra, the infusion of modern technology in American mathematics classrooms has raised doubts and arguments about the future importance of traditional manipulative skills in learning algebra. Some mathematics educators and education researchers suggest that symbolic manipulation skills may retain their importance (Maurer, 1983), while some others believe that "the growing availability of microcomputers and symbolic manipulation software is about to remove much of the rationales for insisting that algebra students attain a high level of competence in symbol manipulation" and that "in the not-too-distant future, all serious users of algebra will do their calculations on computers, just as all serious users of arithmetic rely on calculators now" (Thorpe, 1989, p. 19).

Professional organizations, especially the NCTM, have been making great efforts in bringing different parties into conversations and building compromises and consensus among them. A most recent achievement in this direction is the publication of the *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* (NCTM, 2006). Nonetheless, there is still a long way to go before further achievement can occur. Within the research community, as Star (2005) pointed out, the attention given to these issues has been relatively insufficient. Some of the key notions, such as procedural knowledge and conceptual understanding, still require more careful characterizations in terms of type of knowledge and quality of knowledge. No matter how the skill-understanding priorities are rebalanced, Fey (1989) argued that the interplay between conceptual and procedural knowledge would still be a fundamental issue.

In working with a total of more than 600 prospective elementary teachers at the University of Texas at Austin between 2001 and 2005, I have also noticed some of the issues discussed above. I also have developed a strong interest in better understanding teachers' knowledge and beliefs about the basic algorithms for carrying out mathematical procedures, the alternative algorithms and strategies, and the contrasts and connections among the algorithms and strategies. One of the major goals of my teaching was to find out the extent of future teachers' knowledge, beyond the routines for performing basic arithmetic procedures (such as the four basic operations on whole numbers, integers, and fractions). After assessing this knowledge, I could then help them develop a profound understanding of the underlying rationales, the roles of key algorithms and alternative strategies, and the relationships among various algorithms and strategies. These ideas and efforts have proved to be effective and beneficial to those future teachers, and have also contributed important elements to the conceptual framework for this dissertation research. I will provide more detailed descriptions about this in Chapter 3.

RESEARCH QUESTIONS

In this dissertation research my goal is to study the following two main questions:

(I) What kinds of mathematical knowledge would secondary school algebra teachers draw upon when pondering problem situations that could arise in the teaching and learning of solving algebraic equations?

(II) How may the mathematical knowledge upon which these teachers draw be related to their academic backgrounds and algebra teaching experiences?

Here, a *secondary school* refers to a middle school (6th through 8th grade levels) or a high school (9th through 12th grade levels). An *algebra teacher* is considered as a teacher who has taught first-year secondary school algebra in the last three years or who is currently teaching such a course. *Algebraic equations* simply mean those typical types of equations (linear, quadratic, exponential, rational, etc.) introduced in the secondary school algebra curricula. *Solving* an equation means to find the numerical values of the unknown variable that make the equality a true statement. Among all kinds of strategies, this study will mainly discuss symbolic, tabular and graphical ones. *Problem situations* are written or verbal questions that could be phrased in purely mathematical contexts (e.g., determining the equivalence of two linear equations without solving either of them) or in contexts of teaching and learning (e.g., the advantages and disadvantages of using a certain manipulatives to introduce a method for solving linear equations, evaluating the validity of and the thinking behind a student's solution or responding to a student's statements or questions).

After the literature review (Chapter 2) and the establishment of a conceptual framework (Chapter 3), the two research questions will be specified further in Chapter 4, Research Design. The levels of teachers' mathematics backgrounds and algebra teaching experiences will also be specified.

SIGNIFICANCE OF THE STUDY

The above background analysis provides justification for my selection of the research topic and questions, from both curriculum policy implementation and mathematics teacher education research perspectives.

A major impact of the Algebra for All movement has been the increased requirement in state curriculum policies for students to take algebra courses and pass exit exams. The percentage of secondary school students who have taken algebra courses before graduation has been increasing steadily over the past two decades. The change in policies and its practical influences on students' algebra learning necessitate a highly competitive teaching force with specific qualifications, including having a profound understanding of the algebra topics one teaches. Defining, assessing, and developing such understanding are, therefore, prerequisites for the successful implementation of the current policies.

The proposed study is a direct response to the pressing demands for a better understanding of mathematics teachers' professional knowledge and reasoning. In the past two decades, mathematics education researchers and teacher educators have invested an increasing amount of effort into characterizing and measuring teachers' use of mathematical knowledge in teaching and learning contexts, but more mathematics topics need to be studied before universal theoretical frameworks can be established and generalizable conclusions could be drawn, particularly, at the secondary school level. The lack of systematic, large-scale research on teachers' knowledge of teaching secondary mathematics, including key topic areas such as equations and equation solving, may be partially due to the considerable variance among state and local curriculum standards and instructional materials. With a carefully designed sampling frame, the proposed study could generate information and findings on secondary mathematics topics that may be generalized to a larger scale (e.g., at state level).

Existing studies on mathematics teachers' understanding of the focal mathematics topics, equations and equation solving, are neither abundant nor systematic. The much richer body of literature on algebra learners' understanding of equations and equation solving motivates parallel studies on algebra teachers' conceptions, in that we need to ensure that (a) teachers have conceptual understanding of those concepts and processes in which students commonly make mistakes or have difficulties and that (b) teachers' knowledge for teaching and students' understanding are tied together because a facet of teachers' knowledge for teaching encompasses how students learn specific topics and processes.

The relationship between conceptual understanding and procedural knowledge, the nature and role of mathematical algorithms and procedures, as well as the role of multiple strategies in mathematical problem solving have constantly interested mathematics educators. Researchers also have made recent requests for more thorough investigations on these issues. Studying and developing mathematics teachers' knowledge of algebraic procedures and algorithms, including those related to equation solving, are necessary conditions for making sense of the kinds of teaching practices that could facilitate students' conceptual understanding of those procedures and algorithms.

Chapter 2 Literature Review

INTRODUCTION

By synthesizing research literature and surveying school algebra textbooks, this chapter paves the road for the following two crucial steps in this study:

1. Developing a conceptual framework that is needed for crystallizing the research questions and focal notions, designing instruments, and analyzing data. A cornerstone for building such a framework is an examination of the various approaches researchers have adopted in characterizing the nature and structure of knowledge for teaching school subjects, and a summary of the common features across these characterizations. Meanwhile, as discussed in Chapter 1 (p.9), *mathematical procedures* is chosen as the overarching entity that encompasses equation solving as a representative case and will be integrated into the proposed framework. This concept will be further discussed and defined in the current chapter (pp. 32-33) and Chapter 3 (pp. 71-72).

Specifically reviewed are the following two categories of references:

- Five existing perspectives on teachers' content knowledge for teaching in general or specific to mathematics
- Studies and discussions on the nature and roles of mathematical procedures, algorithms, and procedural knowledge

2. Determining the mathematical topics and issues that should be targeted by this investigation. For instance, areas to be examined include the typical mistakes,

misconceptions and difficulties that have been observed from student learning processes, the various aspects of equation solving which algebra teachers may or may not know well, and the concepts and methods regarding equation solving that are introduced in algebra instructional materials. These become a major source of topics and issues for designing the assessment items and interview questions for this study.

To identify some pivotal topics and issues in learning and teaching equation solving, I review three types of references:

- Mathematics learners' understanding of equations and equation solving
- Mathematics teachers' understanding of equations and equation solving
- Topics related to equation solving in selected Algebra 1 textbooks

PERSPECTIVES ON CONTENT KNOWLEDGE FOR TEACHING

What kinds of content knowledge should or would teachers draw upon for effectively teaching a subject? In the past two decades, education researchers have brought forth a series of perspectives, for teaching practices in general, and for teaching mathematics in particular. These perspectives are based on but beyond empirical observations of teachers' practices and thinking processes, in that they attempt to (a) characterize the nature of or patterns in the phenomena being observed, and to (b) make generalizations, to a certain extent, about the nature and patterns, particularly by providing models that describe components and structures of teacher knowledge. They may need to be validated further by more observations or data from larger scale investigations, in order to be potentially developed into theories on teacher knowledge for teaching (according to Hawking's (1988) definition that a good theory is one that would explain a large class of phenomena and make definite predictions on future observations).

Below, I review five of those perspectives that are most influential, in the sense that each of them (a) was developed collaboratively through teacher education research and practices, (b) has been disseminated through academic presentations and publications and (c) has been frequently referenced by other researchers.

Leinhardt and Smith: Content Knowledge Underlying Teaching Expertise

As one of the early contributions to research on knowledge for teaching, Leinhardt and Smith (1985) explored the organization and content of expert arithmetic teachers' subject matter knowledge of fractions. They pointed out that previous work in the field of expert performance mainly attempted to analyze the structural or procedural aspects which were independent of domain-specific content knowledge, and few studies had examined the types and levels of subject matter skills used by teachers. As a result, they believed that the exploration of domain-specific knowledge must be undertaken.

The authors considered teachers' *subject matter knowledge* as the most fundamental source of the cognitive aspects of expertise in teaching and defined it as including "concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation" (p. 247). To the authors, subject matter knowledge influences and is intertwined with another major source of teachers' expertise, *lesson structure knowledge*, which includes "the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly" (p. 247). In other words, subject matter knowledge acts as a resource in the selection of examples, formulation of explanations, and use of demonstrations.

The authors' in-depth analysis of expert teachers' explanation behavior in teaching fractions revealed wide variations in their knowledge as well as substantial differences in the details of their presentations to students. Some teachers displayed relatively rich conceptual knowledge of fractions, while others appeared to rely on precise knowledge of algorithms. There were considerable differences in the level of conceptual information and procedural algorithmic information presented. Teachers had decidedly different emphases in their presentations, and they entered the topics differently. The authors also noticed the differential use of representation systems, such as the number line, the area model, and numerical forms.

Although the subject matter being studied here is an elementary mathematics topic, the authors have made analyses and inferences about knowledge for teaching in a general manner with the "hope that a detailed analysis of expert teachers' fraction knowledge will shed light on how knowledge is used in effective teaching" (p. 249).

The authors' descriptions of the types of teacher knowledge suggest that they come from both the positivist and the behaviorist traditions because they seem to view teacher knowledge as "a set of lawlike generalizations that can be identified through classroom research and applied by practitioners" (Calderhead, 1996, p. 715) (positivist), and also to identify teacher knowledge and expertise from observable skills, behaviors, emphases, and preferences (behaviorist). While each knowledge category they defined is central to the associated types of subject matter skills, other kinds of knowledge could likely be involved and may not be neglected. For instance, lesson structure knowledge is at the heart of "the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly" (Leinhardt and Smith, 1985, p. 247), but other types of knowledge (such as knowledge of students' cognitive development and prior learning experiences) could be also important for lesson planning and the aforementioned skills.

Shulman: Teachers' Content Knowledge for Teaching

In his Presidential Address at the 1985 meeting of the American Educational Research Association, Shulman criticized a missing paradigm in educational policy, research, and practice at that time, which he referred to as the tendency of overlooking subject matter while focusing solely on the pedagogical aspects in defining teacher effectiveness (Shulman, 1986a, 1986b, 1987). Based on the research program that he and his colleagues were operating, *Knowledge Growth in Teaching*, Shulman introduced "a more coherent theoretical framework" for probing "the complexities of teacher understanding and transmission of content knowledge" (1986a, p.9). The framework distinguishes among three types of content knowledge:

1. Subject matter content knowledge

It is "the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986a, p.9) and involves two types of structures of the subject: *substantive structure* (the ways in which the basic concepts and principles of the discipline are organized to incorporate its facts) and *syntactic structure* (the ways in which truth and validity are established).

2. Pedagogical content knowledge

This is “the particular form of content knowledge that embodies the aspects of content most germane to its teachability” (Shulman, 1986a, p.9). It is knowledge about “the ways of representing and formulating the subject that make it comprehensible to others” and also includes “an understanding of what makes specific topics easy or difficult for a certain group of learners” (1986a, p.9)

3. Curricular knowledge

The third type of knowledge is of the variety of instructional materials available to programs that are designed for the teaching of the subject and topics at a given level, as well as the characteristics of using certain materials in certain circumstances.

The notion of pedagogical content knowledge highlighted the nature and content of teachers’ subject matter understanding in ways that previous focuses on teacher credentials and pedagogical behaviors had not. It has been recognized as a special form of content knowledge that is characteristic of teachers and, ever since, has led to a completely new trend in research on teacher knowledge across the mathematics, English, language, history, and social studies disciplines.

Shulman’s perspective on teachers’ content knowledge for teaching seems to be acquisitionist: Such knowledge is a cognitive entity that can possibly be acquired and possessed by teachers, and its three basic components can be identified and distinguished. Correspondingly, a transmission model on teaching and learning underlies his notion:

“Teachers understand a subject and, through appropriate tasks, explanations, and demonstrations, develop this understanding in children” (Calderhead, 1996, p. 716). Such a perspective and its related approaches to research, however, leave gaps in efforts to fully understand the mathematical demand of teaching. Ball and her colleagues reveal that one of the gaps

> centers on the remaining distance between studies of teacher knowledge and teaching itself. To understand the mathematical work of teaching would require a closer look at practice, with an eye on the mathematical understanding that is needed to carry out the work. (Ball, Lubienski, & Mewborn, 2001, p. 449)

Ball, Bass and Colleagues: Mathematics Knowledge in and for Teaching

During the past decade, Ball, Bass, and some other researchers have hypothesized that pedagogical content knowledge as an acquired form of expertise may not always equip teachers with the flexibilities needed to manage the complexities of and uncertainties in the dynamic teaching practices (Ball & Bass, 2000). They have been attempting to answer what mathematical knowledge is entailed in teaching and how to assess it (Ball, 1999, 2000; Ball & Bass, 2000; Ball, Bass, Hill, & Schilling, 2005; Ball, Lubienski, & Mewborn, 2001; Hill, Schilling, & Ball, 2004; Kennedy, Ball, & McDiarmid, 1993) and have devoted themselves to developing a *practice-based theory of mathematical knowledge for teaching* (Ball & Bass, 2003, Stylianides & Ball, 2004). By practice-based they view mathematics teaching as involving substantial mathematical work and focuses on the kind of mathematics that emerges within the core domains of teaching tasks:

- Choosing or designing mathematically appropriate and comprehensible definitions, examples, and tasks; representing mathematical ideas and making them available to students
- Giving mathematically appropriate and accurate explanations and justifications
- Posing mathematically sound questions and problems to probe, assess, or promote student learning
- Attending to, interpreting, and evaluating students' questions, solutions, problems, and insights (both predictable and unusual), and responding productively
- Making judgment about the mathematical quality of instructional materials and modifying as necessary; appropriately sequencing content and problems
- Establishing and managing mathematical discourse

Along this direction several mathematical concepts and processes have been studied, such as even numbers, multi-digit subtraction, multiplication of integers and decimals, fractions, definitions of polygons, and reasoning and proof. Two major features of mathematical knowledge that is useful in teaching have emerged from these studies (Ball & Bass, 2003): (a) such knowledge must be unpacked, and (b) it must be connected, both across mathematical domains at a given level and across time as mathematical ideas develop and extend.

Instead of decomposing knowledge for teaching into components, Ball and her colleagues began with describing the typical pedagogical scenarios in which knowledge for teaching mathematics may be needed or used. It acknowledges the complexity and richness of teaching practices and emphasizes the high mathematical demands in mathematics teaching. The scenarios or contexts provide entry points for researchers potentially to access and analyze the mathematical knowledge involved in teaching. Such conceptualization is more aligned with the theory of situated cognition which considers teacher knowledge as "lying within the interaction of particular contexts and situation" and "as much dependent on the environment within which teachers work as on the individuals" (Calderhead, 1996, p. 716).

In the past three years, Ball, Bass and their research team also have followed Shulman's categorical scheme and proposed refined models for knowledge of mathematics for teaching. Their latest model (Figure 2.1) (Ball, Bass, Hill, & Schilling, 2005; Ball, Bass, Hill, & Thames, 2006) breaks subject matter knowledge into two categories: *Common Content Knowledge*, which can be developed in anyone who has had school mathematics education, and *Specialized Content Knowledge*, which is used mainly by teachers. Meanwhile, the model makes a distinction between two main categories in pedagogical content knowledge: *Knowledge of Content and Students* and *Knowledge of Content and Teaching*. This model highlights the kind of mathematical content knowledge that is the specialty of teachers, and acknowledges that knowledge of mathematics for teaching is partially the product of content knowledge interacting with students in their learning processes and with teachers in their teaching practices. What remains for researchers is to (a) find representative cases of teachers' specialized mathematical content knowledge; (b) provide refined characterizations of the categories; and (c) demonstrate the various interactive processes through which teachers' knowledge of the content, knowledge of student learning, and knowledge of teaching are blended and connected.

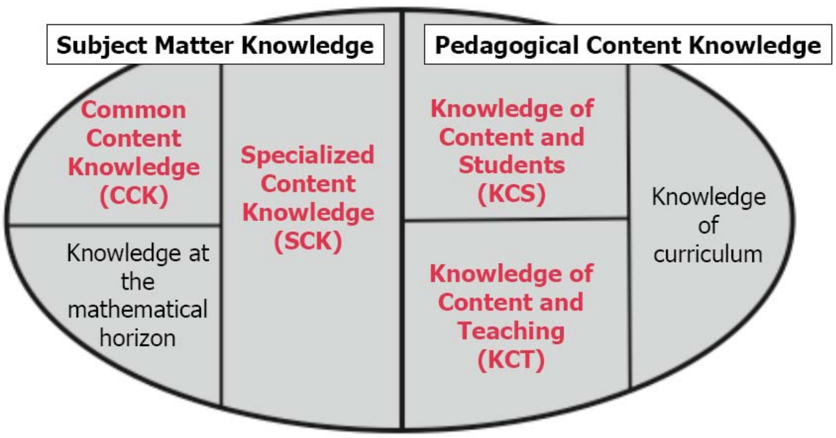


Figure 2.1 A model of mathematical knowledge for teaching

The three frameworks discussed thus far are conceptualizations of knowledge for teaching (or teaching mathematics) in general. They may not be able to reflect the characteristics of teachers' knowledge use in teaching specific mathematical subjects, topics, concepts, methods, or mathematical ways of thinking and reasoning. Below, I summarize two frameworks that focus particularly on the subject of school algebra.

Artigue and Colleagues: Teachers' Professional Knowledge of Algebra

In characterizing and measuring teachers' professional knowledge in algebra and the way it develops, Artigue and her research team have launched several research projects that were guided by a Multidimensional Grid for Professional Competence in Elementary Algebra (MGPCA) (Artigue et al, 2001; Doerr, 2004). The design of the grid was based on two hypotheses: (a) teachers' professional development in algebra is "a complex genesis that relies on a mixture of competencies specific to this domain and more transversal competencies which strongly intertwine in practice" (p. 25), and (b) the knowledge that underlies specific competencies influences different professional decisions "in a non-uniform way and that it cannot necessarily be made explicit" (p. 26).

The grid is structured around three dimensions of knowledge which, as the authors explicitly stated, are *non-independent*:

1. The epistemological dimension

This dimension is about knowing (a) the complexity of the algebraic symbolic system and the difficulties of its historical development, (b) how to use flexibly algebraic tools in solving different kinds of problems that are internal or external to the field of mathematics, (c) the content and structure of algebra, (d) how to cope with algebraic objects by taking into account both their semantics and syntax, (e) the role and place of algebra in mathematics, (f) the nature of valuable tasks for learners, and (g) the connections between algebra and other areas of mathematics and physical phenomena.

2. The cognitive dimension

This dimension deals with knowledge about learning processes in algebra, which includes knowing (a) the development of students' algebraic thinking, (b) students' interpretations of algebraic concepts and notation, (c) students' misconceptions and difficulties in algebra, and (d) ways to motivate learners, etc.

3. The didactic dimension

This dimension involves knowledge of (a) the algebra curriculum, (b) the specific goals of algebraic teaching at a given grade, (c) possible progressions and activities for the teaching of algebra, (d) well-adapted assessment tasks, and (e) algebra-related educational resources, such as textbooks, other materials, websites, and computer tools.

Because this framework was developed to characterize the *professional knowledge basis* for teaching algebra, it is quite broad and not necessarily focused on the *mathematical* knowledge for teaching algebra. Also, it depicts teacher knowledge more as entities that a teacher has acquired or is to acquire, than about the processes and contexts in which a teacher develops such knowledge. Nonetheless, the recognition of and distinction among the three dimensions still provide enlightenment: (a) the epistemological dimension is about teachers' knowledge of algebra as a socially and culturally developed mathematics subject, (b) the cognitive dimension is about teachers' knowledge of student learning of algebra, and (c) the didactic dimension is about knowledge of how to teach algebra to students. In one way or another, we can find that these dimensions are reflected in many other frameworks on knowledge of mathematics for teaching. When needed, we can also customize each of the three dimensions to describe more specific content areas and topics in school algebra.

Ferrini-Mundy and Colleagues: Knowledge of Mathematics for Teaching Algebra

Since 2001 Ferrini-Mundy and her colleagues at Michigan State University have been working on two consecutive National Science Foundation grants, *A Study of Algebra Knowledge for Teaching at Secondary Level* (2001-2004) and *Knowing Mathematics for Teaching Algebra* (2004-2007). They have presented their work in various venues (Burrill, Ferrini-Mundy, Senk, & Chazan, 2004; Ferrini-Mundy & Burrill, 2001; Ferrini-Mundy, Senk, & Schmidt, 2004). One of the major publications of the first project includes a conceptual framework for characterizing knowledge for teaching school algebra (Figure 2.2) (Ferrini-Mundy et al., 2003); and a framework for designing items to assess knowledge for teaching algebra (Figure 2.3) (Floden, Ferrini-Mundy, McCrory, Senk, & Reckase, 2005). Both of them become the foundation for the second project.

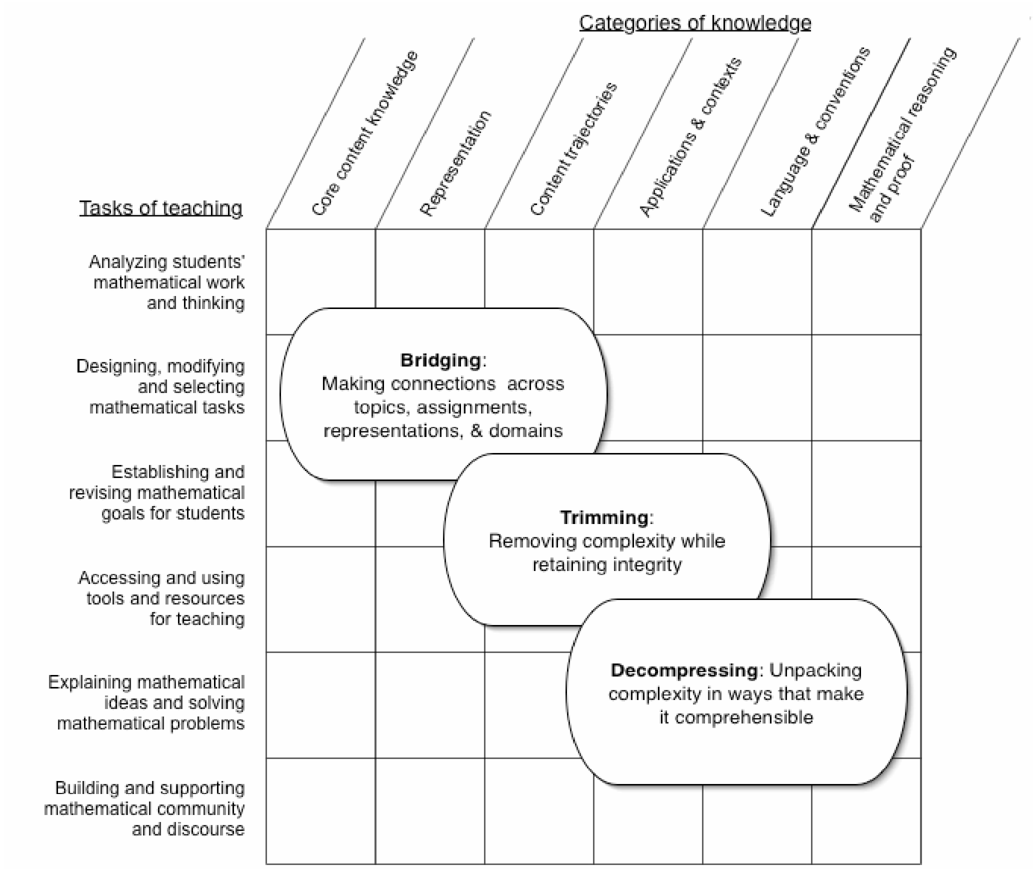


Figure 2.2 A framework characterizing knowledge for teaching school algebra

In Figure 2.2, the horizontal dimension of the grid indicates the fundamental *categories of knowledge* involved in teaching algebra, and the vertical dimension identifies several *tasks of teaching*, in which teachers may apply their mathematical knowledge. The three overarching categories, *decompressing*, *trimming*, and *bridging*, are more sophisticated mathematical practices that utilize multiple elements of knowledge for teaching algebra and involve multiple tasks of teaching.

The framework illustrates the overall landscape of knowledge for teaching algebra: which major types of knowledge may be used and in which contexts they may be used. The various combinations of categories of knowledge with tasks of teaching, plus the three overarching categories, demonstrate the vast complexity of knowledge use in practice.

For the purposes of characterizing, distinguishing, and measuring different kinds of knowledge for teaching algebra in a subtler and more in-depth way, the researchers also have created a three dimensional construct, as illustrated in Figure 2.3 below.

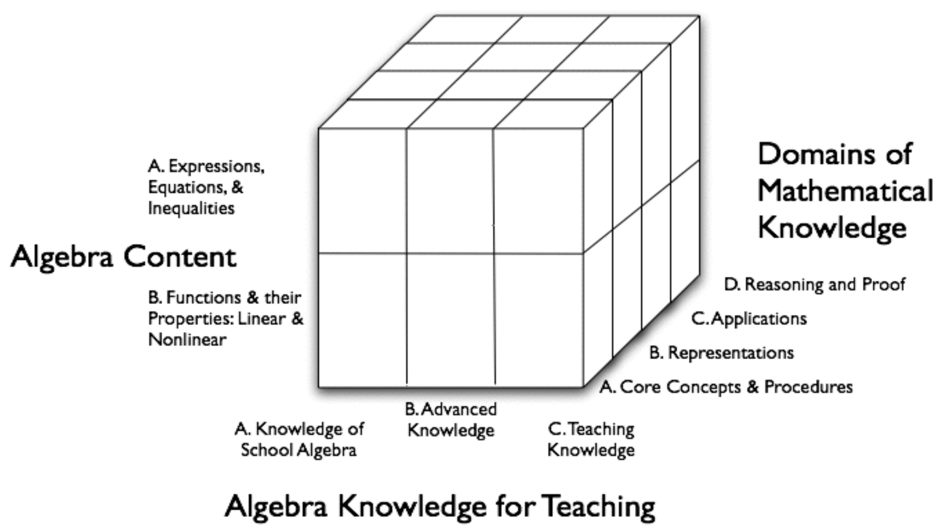


Figure 2.3 A framework for assessing knowledge for teaching algebra

In Figure 2.3, the base of the matrix consists of three types of *algebra knowledge for teaching* (knowledge of school algebra, advanced mathematical knowledge, and teaching knowledge on the X-axis) and four *domains of mathematical knowledge*, or aspects of algebra teaching and learning (core concepts and procedures, representations, applications and contexts, and reasoning and proof on the Y-axis). The Z-axis contains two major themes in school *algebra content*: expressions, equations and inequalities, and functions and their properties. Assessment items can be specifically written for each cell in the matrix, for instance, knowledge of school algebra that is related to a core procedure for solving equations. In other words, each assessment item would be uniquely located in Figure 2.3 as a coordinated system. A balanced selection of items could then be used to evaluate teachers' knowledge for teaching in a particular dimension and, eventually, contrast or relate such knowledge among all three dimensions.

In the above framework, the categorization on the X-axis (algebra knowledge for teaching) is most relevant to the other frameworks that have been discussed so far, in the sense that they all attempt to identify major components of mathematics knowledge for teaching. Particularly, through comparing the X-axis with the framework developed by Artigue and her colleagues, I realized that (a) if we combined *knowledge of school algebra* and *advanced mathematical knowledge*, these two categories would become a major part of the Epistemological Dimension in Artigue's framework, and (b) by the definition of the *teaching knowledge* category, such knowledge could be broken down into two aspects to reflect the Cognitive Dimension and Didactic Dimension in Artigue's framework. The definition of the *teaching knowledge* category to which I have just referred is as follows:

> . . . it includes mathematical knowledge specific to teaching algebra that may not be taught in advanced mathematics courses or known by some mathematicians. It includes such things as what makes a particular concept difficult to learn and what misconceptions lead to specific mathematical errors. It also includes mathematics needed to identify mathematical goals within and across lessons, to choose among algebraic tasks or texts, to select what to emphasize with curricular trajectories in mind, and to enact other tasks of teaching. (Ferrini-Mundy et al, 2003, p. 12)

Such a triadic categorization of mathematical knowledge for teaching laid the foundation for me to structure my own conceptual framework, which will be discussed in the next chapter.

In summary, each of the five perspectives reviewed above contains a propositional model (or framework) for characterizing the structure and major components of teachers' content knowledge for teaching. The categorical representations of teacher knowledge for teaching helped me to identify a few common, core elements upon which to build my own framework for the study (details in Chapter 3). Although, labeling and studying categories of teacher knowledge have a potential pitfall: we might start to view the categories as abstract and isolated entities that a teacher can acquire, possess, and apply in different teaching contexts (Borko & Putnam, 1996). The comprehensive conceptualizations by Ball and colleagues, as well as by Ferrini-Mundy and colleagues, emphasize the teaching tasks in which teacher knowledge is applied, which reminds us that the nature and use of teacher knowledge are very sensitive to the relevant teaching contexts and tasks.

MATHEMATICAL PROCEDURES, ALGORITHMS, AND PROCEDURAL KNOWLEDGE

Mathematical Procedures and Algorithms

In mathematics education, the term *mathematical procedure* is not rigorously or universally defined. It has been used to denote two related but distinct types of mathematical entities. One type of entity is mathematical operations and processes, such as counting, computing with the four basic operations, transforming algebraic expressions (combining or expanding terms), solving equations, graphing functions, constructing geometric objects with straightedge and compass, calculating limits, and finding derivatives and integrals of functions. Such use assumes the broad meaning of procedures. The other type of entity is the *rules, routines,* or *algorithms* employed in carrying out those operations and processes of the first type. For instance, one may use the factoring method, the completing the square method, or the quadratic formula to solve a quadratic equation.

In either case, a mathematical procedure is often not explicitly defined and it is left for the readers to determine the exact meaning based on either common usage or the context. For example, Hiebert and Lefevre (1986) provided the following characterization of procedural knowledge:

> . . . as we define it here, [it] is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics . . . The second part . . . consists of rules, algorithms, or procedures used to solve mathematical tasks. (p. 4)

How the word procedures is used in the above description seems to fall into the second type of use, since procedures are considered as being in the same family as rules and algorithms. Battista (1999) mentioned various kinds of procedures: *arithmetic, algebraic, symbolic, computational, rote,* and *invented*. These seem to be a mixture of both types of use.

In contrast, there are widely agreed upon definitions for *algorithms*. Kilpatrick, Swafford and Findell (2001) quoted Knuth (1974), who defined an algorithm as a "precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps" (p. 33). Such definition reveals four basic features of an algorithm: (a) it is a well-defined sequence, with (b) input, (c) output, and (d) has finite steps.

I see two kinds of connections between procedures and algorithms. First, in a broad sense (which corresponds to the first type of use indicated above), all algorithms can be called procedures. Such connection may have been exactly one of the causes for the interchangeable uses of the two terms. Second, in a narrower sense (which corresponds to the second type of use), a procedure is a form or title, while an algorithm is a collection of specific actions or strategies that are formulated for carrying out a particular procedure. Both of these connections were mirrored in the definition given by Mingus and Grassl (1998): "An *algorithm* is a computational recipe for the systematic execution of a *procedure* designed to solve a specific problem"(p. 34) that maintains a list of characteristics.

In this study, I adhere to the first type of use, and define mathematical procedures as mathematical operations and processes that act on initial inputs and produce desired results. My definition of procedures will be clarified further in Chapter 3 when I establish the conceptual framework.

The Features and Roles of Procedures and Algorithms

In carrying out a certain mathematical procedure, multiple algorithms or strategies could be used, and different representations could be involved (e.g., written symbols, concrete objects, fingers, slide rules, visual diagrams, and mental images). Existing literature (e.g., Campbell, Rowan & Suarez, 1998; Kilpatrick, Swafford, & Findell, 2001; Usiskin, 1998) has discussed the following key measures for evaluating standard and alternative algorithms and strategies associated with a given mathematical procedure:

1. *Validity* This is the most basic criterion for a strategy to be considered and adopted. Given opportunities, students may invent their own strategies based on observations or guesses for one or two specific cases, but such strategies may not be valid for most other cases.

2. *Accuracy* (or *precision*) An algorithm must be completely accurate or precise. Algorithms and strategies that involve the use of concrete objects or pictorial representations often yield approximated outputs (e.g., solving a linear equation by using algebra tiles or by tracing the corresponding line graph).

3. *Generality* (or *generalizability*) For a given procedure, a fundamental difference between the standard algorithm and some of the alternative algorithms and strategies typically lies in the range of cases to which the algorithms and strategies can be applied. In solving quadratic equations, for instance, the completing the square method and the quadratic formula are general algorithms that apply to any quadratic equation,

while the factoring method would work only for a small amount of equations with integral coefficients.

4. *Efficiency* The efficiency of an algorithm may depend on who is using it and on the nature of the problems. The standard algorithm for adding two multi-digit whole numbers (starting with the lowest digit, moving to the left toward higher digits, carrying one when the sum is larger than 10) uses less writing than the alternative strategy of Partial Sums (adding each digit separately, then putting the sums together), but it may not be as efficient as using the method of Compensation mentally (e.g., $29 + 37 = (30 - 1) + 37 = (30 + 37) - 1 = 67 - 1 = 66$).

In theories on advanced mathematical computations and computer programming, the efficiency of an algorithm could be measured by the time it takes to solve, the number of basic operations it involves, or the memory it occupies. It is much less straightforward with school mathematics. Star (2005) argued that there are no absolute criteria for measuring efficiency because different users may have distinct experiences with various algorithms. My stance is that efficiency is a relative term, and, similar to the case of advanced mathematics computations and computer programming, it should be independent of individual user, computer, or problem. In other words, it is a collective and statistical notion. There could be a set of reasonable criteria upon which mathematics educators and education researchers might widely agree. For instance, an algorithm is relatively efficient if the majority of users think so after using it to solve a variety of problems.

5. *Transparency* Teachers, students, and others may hope that the algorithms and strategies they use are intuitive and easy to understand. Unfortunately, there seems to be a dilemma between transparency and some other features, such as efficiency and generality. The more an algorithm is general or efficient, the fewer steps it may take from input to output or the less it may consider the details and characteristics of specific cases. As a result, it is often less transparent. A good example is the contrast between the quadratic formula and the method of completing the square. The first one is a shortcut or compact version that is directly derived from the second one. They are equally general, but the first one is definitely more efficient, and the second one is more transparent.

In summary, the major characteristics of standard algorithms include their absolute validity and accuracy, and the highest level of generality. Compared with alternative algorithms, however, they may not always be as efficient or transparent. To be effective in teaching multiple algorithms and strategies and to help students see the strengths of these, mathematics teachers themselves need to have a solid knowledge background.

Algorithms are important in school mathematics because they encapsulate important mathematical facts, can help students understand better the fundamental operations and concepts, and pave the way for learning more advanced topics (Kilpatrick et al, 2001). They can bring automaticity, speed and reliability to the completion of mathematical procedures, and can reveal subtle relationships between given information and answers to problems. Algorithmic thinking is a powerful reasoning method for analyzing and solving mathematical problems, constitutes the foundation for computer programming, and helps learners to make the transition from arithmetic to algebra (Driscoll, 1999).

Meanwhile, mathematics educators and researchers have also raised concerns over some potential dangers inherent in the teaching, learning, and use of algorithms (Kamii & Dominick, 1998; Usiskin, 1998). The list that follows addresses some of these concerns:

1. Because the written representations of algorithms are procedural and often in packed forms, they may conceal certain important information and facts. For example, the standard algorithms for the four operations make the concepts of base-10 and place value almost invisible.

2. Teachers may introduce the algorithms as established routines for students to memorize and practice without revealing the underlying rationale and connections with other mathematics concepts and procedures, or engaging students in exploring with their own thinking and reasoning.

3. After applying certain algorithms, students may be easily satisfied with the results they get and, hence, blindly accept them without checking for validity or reasonableness.

4. Students may over-apply a general algorithm to all situations without considering the nature of special cases and not, correspondingly, switching to appropriate alternatives.

5. Some students may become overly dependent on certain algorithms such that they cannot complete a procedure if some required conditions (e.g., paper and pencil, calculators) of those algorithms are missing.

To examine further the nature of equation solving as a mathematical procedure involving algorithms and strategies, I have also reviewed studies on procedural knowledge and its relationship with conceptual understanding.

Reconceptualizing Procedural and Conceptual Knowledge

For decades, the nature and roles of procedural and conceptual knowledge, as well as the relationships between them, have been the central issues for discussions among mathematics educators, education researchers, and cognitive psychologists (Hiebert & Lefevre, 1986; Silver, 1986). Previous studies have demonstrated the mutually dependent and interwoven relationship between skills and understanding. On the one hand, solid procedure skills can trigger the development of new concepts (Baroody & Ginsburg, 1986) and facilitate the application of conceptual knowledge by reducing the mental effort required in problem solving (Case, 1985; Kotovsky, Hayes, & Simon, 1985). Lack of skills or using incorrect skills for a few years can render instructions that emphasize conceptual understanding less effective (Resnick & Omanson, 1987). On the other hand, when students have learned skills without understanding, it can be difficult to engage them in activities aimed at conceptual understanding (Mack, 1995; Rittle-Johnson & Alibali, 1999; Wearne & Hiebert, 1988), and they can typically do no more than apply the learned procedures, whereas students who learned procedures with understanding can modify or adapt them to make them easier to use (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998).

In reality, however, procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics (Kilpatrick et al, 2001), which, for some educators (Brownell, 1987; Wu, 1999), has created a false and harmful dichotomy. The disagreement may have its root deep in the lack of consistent categorizations and characterizations of procedural knowledge and conceptual knowledge. For instance, in Hiebert and Lefevre (1986), as quoted earlier, the authors defined conceptual knowledge as “knowledge that is rich in relationship” and as “a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information,” (pp. 3-4) while they defined procedural knowledge as “made up of two distinct parts. One is composed of the formal language, or symbolic representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks” (p. 4). Here, conceptual knowledge is defined in terms of the quality or level of complexity of one’s knowledge – richness of the inter-connections – but procedural knowledge is defined by the content of one’s knowledge – knowledge of mathematical procedures. Such entanglement of knowledge quality and content makes the uses of the terms problematic, especially for procedural knowledge (De Jong & Ferguson-Hessler, 1996; Star, 2000).

As a consequence of misuse and misinterpretation of terminologies, as well as the prioritization of conceptual knowledge over procedural knowledge, the mathematics education research community has been given insufficient attention to learners’ acquisition of procedural skills. By searching the Educational Resources Information Center (ERIC) database, Star (2005) reported that the number of mathematics education journal articles using the terms *conceptual knowledge* or *conceptual understanding* was approximately four times that of journals using the terms *procedural knowledge* or *procedural skill*. Similarly, in the last decade, only six articles published in the Journal for Research in Mathematics Education (JRME) included procedure or algorithm as a keyword. And out of the roughly 100 articles related to the development of students' mathematical content knowledge, only 11 carefully investigate students' knowledge of procedures.

The research report synthesized by Kilpatrick et al. (2001) recognized mathematical proficiency as consisting of five interwoven and interdependent strands: *conceptual understanding*, *procedural fluency*, *strategic competency*, *adaptive reasoning*, and *productive disposition*. In these five phrases, the adjectives are all used as descriptors of the high quality of a certain proficiency, and, therefore, the definitions are more consistent. Take the first two as an example. Conceptual understanding refers to "an integrated and functional grasp of mathematical ideas" (p. 119) and procedural fluency refers to "knowledge of when and how to use procedures appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 123).

Besides studying students' knowledge of mathematical procedures and algorithms, researchers have also examined mathematics teachers' knowledge in this respect. For example, in her comparative study, Ma (1999) concluded that Chinese teachers' knowledge systems are much more coherent than those of American teachers, in the sense that the Chinese teachers “not only know how to carry out an algorithm but also know why it makes sense mathematically,” “approach problems with not only the standard algorithms but also alternative strategies, and find the optimal solution,” and “understand the relationship among fundamental mathematics concepts and procedures, and make connections” (pp.107-113).

STUDENTS’ UNDERSTANDING OF EQUATIONS AND EQUATION SOLVING

Procedural and Structural Views of Equations and Equation Solving

Mathematically, equality of numerical expressions (e.g., $3 + 5 = 2 + 6 = 2 \times 4$) represents an equivalence relation that satisfies the reflexive, symmetric, and transitive properties. A common preconception held by elementary and middle school students is that the equal sign stands for a unidirectional command for action (or operation) that leads from numbers or symbols on the left side toward the result on the right (Behr, Erlwanger, & Nicols, 1976). Consequently, some young learners may record 12 as the answer for $7 + 5 = \_\_\_\_ + 9$ and some others may not accept an arithmetic or algebraic expression (e.g., $3 + 4$, or $7x – 5$) as a meaningful object because they believe that two elements connected by an operation have to be closed by the equal sign and followed by a result (Herscovics & Linchevski, 1994). In performing multi-step arithmetic or algebraic computations, many students tend to use equal signs to combine a sequence of operations into one line (e.g., $4 + 5 = 9 + 7 = 16 – 5 = 11$) (Vergnaud, 1979).

Such limited notion of the equal sign in the arithmetic framework may likely hinder students from making sense of algebraic equations in later studies (Booth, 1984; Mevarech & Yitschak, 1983). A typical equation, such as $3x + 5 = 11$, can perfectly fit into a student's existing notion of acceptable equations, while $3x + 5 = 2x$ may not. Filloy and Rojano (1984) referred to an *arithmetical* equation as having the form $ax + b = c$ and an *algebraic* equation as having the form $ax + b = cx$. They claimed that one can find, here, the exact delineation between arithmetic and algebra, a so-called *didactic cut*, when learners switch from arithmetical to algebraic equations.

Kieran (1992) examined the above phenomenon and defined it as the dichotomy between *procedural* and *structural* views of equations (and of algebra, in general). Prior to the study of linear equations, students would find the solution to $2 \times \_\_\_ + 1 = 5$ with a procedural approach (i.e., trying various numerical values until the arithmetic operations on the left could yield 5 on the right). Kieran noted that, "At this stage of a learner's mathematical education," when a student starts to deal with the algebra form $2x + 1 = 5$, "the introduction of formal algebra requires a teacher-directed move away from a procedural approach towards a structural approach" (p. 392). Here, *structural* refers to "a different set of operations that are carried out, not on numbers, but on algebraic expressions" (p. 392). For example, the solution to $3x + 2 = 15$ can involve the step $3x + 2 - 2 = 15 - 2$, which has nothing to do with either the final solution or any numerical instantiation. Learners must grasp the structure of an equation if they are expected to be successful in solving equations of the form $ax + b = cx$. In helping learners make smoother transitions, researchers and educators have designed and experimented with certain strategies, such as letting students construct arithmetic equalities that have different and multiple operations on the two sides (Herscovics & Kieran, 1980; Kieran, 1979, 1982).

Three Types of Equivalences Involved in Equations and Equation Solving

When students learn to solve linear equations, they have to handle three distinct types of equivalence relations:

1. Equivalence involved within equations that link variable expressions (e.g., $2x + 1 = 7 - x$). In such cases, the equal sign still denotes an equivalence relation on the set of all expressions in the same variable, but only for the numerical value of the variable for which the equalities hold. For example, the three expressions $2x + 1$, $7 - x$, and $3x - 1$ are in the same equivalence class only when $x = 2$. If such a value of the variable has not been specified, then the properties of an equivalence relation may not apply. For example, for two unrelated equations $2x + 1 = 7 - x$ and $7 - x = x + 5$, one cannot conclude $2x + 1 = x + 5$ based on the transitive property, because the first two equalities hold for different values of the variable ($x = 2$ and $x = 1$, respectively).

2. Equivalence among algebraic expressions that are connected by transformations which preserve values (e.g., $2x + 3x = x + 4x = 5x$). Each equation in such cases is an identity that holds for all allowable values of the variable. Algebra students experience this type of equivalence when they learn how to operate on expressions, such as simplifying an expression by combining like terms.

3. Equivalence between the original equation and those derived in the solving process. By the standard definition, two equations are equivalent when they have exactly

the same solution set. The equivalent classes created by this definition would be huge and structureless. It makes more sense to focus on the kind of equivalent equations that is linked by solution-preserving transformations. In school algebra, equivalent equations are typically generated through two basic types of transformations: (a) Replacing an expression in an equation with an equivalent expression (for example, the equation $5x + 3 = 3x + 7 – x$ becomes $5x + 3 = 2x + 7$), and (b) Performing the same operations on both sides. The expressions on each side are not necessarily equivalent (for example, after 3 is subtracted from both sides, the equation $5x + 3 = 2x + 7$ becomes $5x = 2x + 4$; $5x + 3$ is not equivalent to $5x$ and neither is $2x + 7$ equivalent to 3).

Greeno (1982) found that many algebra students are not aware of these distinctions. They do not realize that it is only the correct solution that will yield equal values for the two sides of an equation. Kieran and Sfard (1999) showed that many students just follow the rules of transforming expressions into equivalent ones without understanding the underlying properties or the meaning of the equality (equivalence) between two equivalent expressions, such as $3(x + 2) = 3x + 6$.

Students are taught procedural steps for solving equations, but the concept of equivalent equations is often not explicitly emphasized by algebra textbooks or by algebra teachers (Steinberg, et al, 1990). As a result, when a group of 8th and 9th graders were provided with the definition of equivalent equations (i.e., equations that have the same solutions) and asked to determine and justify the equivalence of a list of equation pairs, even though most of them knew how to use transformations to solve simple equations, only about 40% of them made judgments based on the transformations that connected the equation pairs. About 30% still had to solve both equations in order to determine the equivalence.

Solving Equations: Formal vs. Alternative Approaches

Kieran (1992) identified a list of typical strategies used by children in solving linear equations:

1. Using number facts (e.g., $3 + x = 5$; $5 – 3 = 2$; therefore $x = 2$)
2. Using counting techniques (e.g., $3 + x = 9$, counting from 3 to 9 leads to $x = 6$)
3. The cover-up method (e.g., If we cover th9 in $2x + 9 = 5x$, it becomes $2x + ? = 5x$, So "?" must be equal to $3x$, (i.e., $3x = 9$). Then we cover the x in $3x = 9$, which gives us $3 \times ? = 9$. The "?" should equal 3; therefore $x = 3$.)
4. The undoing (or working backwards, or back-tracking) method (e.g., We view the equation $2x + 4 = 18$ as beginning with x, multiply it by 2, then add 4, and get 18. We can then undo the above operations in reverse order. Subtracting 4 from 18, we get $2x$ (i.e., $14 = 2x$). Finally, dividing 14 by 2, we get $x = 7$.)
5. Trial and error substitution
6. The method of transposition (change-side change-sign) (e.g., In $2x + 4 = 18$, we move 4 from the left side to the right side and change its sign; the equation becomes $2x = 18 + (-4)$.)
7. Using formal rules (performing the same operation on both sides, i.e., the balancing method)

Between the 1970s and the 1990s, researchers carried out various teaching experiments and studies on how secondary school students solve linear equations.

Whitman (1976) studied the relationship between the cover-up and the formal procedure in a teaching experiment with 7th graders. She found that students who learned to solve equations by means of only the cover-up method performed better than those who learned both ways, whereas those who learned only the formal procedures performed worse than those who learned both strategies. This suggests that students who are taught the formal method alone are not conceptually prepared to approach equations as mathematical objects with formal, structural operations.

Petitto (1979) referred to the first five methods in Kieran's list as intuitive and pointed out the fact that the methods typically do not generalize. In her study of 9th grade algebra students, she found that those who used a combination of formal and intuitive processes were more successful than those who used only one of these methods.

Kieran (1984, 1988, 1989) has conducted a series of research studies regarding the last two methods, transposition and formal rules. In a teaching experiment carried out with 12-year-old beginning algebra students (Kieran, 1984), one particular error was detected and named the *redistribution error*: some students transformed an equation like $x + 37 = 50$ into $x + 37 - 10 = 50 + 10$, so that the subtraction of 10 from the left side was balanced by the addition of 10 on the right side. The students who committed this error were those who preferred the change-side-change-sign method at the beginning of the study. They seemed to be confused by these two methods.

In contrast, the students who did not commit this error were those who preferred trial-and-error substitution at the beginning of the study. That is, they had a better sense of keeping an equation in balance.

Through an experiment with 7th grade students, Kieran (1988) found that the subjects who began the study with a preference for the undoing method were unable to make sense of the formal rule method, while those who had an initial preference for the trial-and-error substitution method and who viewed an equation as a balance achieved better understanding.

The method of transposition can be considered a shortened version of the formal rule method. A cognitive difference between them is that the formal method emphasizes the symmetry of an equation, while such emphasis is absent from the transposition method. Consequently, they appeared to be perceived quite differently by beginning algebra students (Kieran, 1988; 1989).

Herscovics and Linchevski (1994) gave a class of students 50 linear equations to solve before the formal algebraic method was taught. They found that when the unknown appeared more than once on the same side of an equation (e.g., $3n + 4n = 45$) or on both sides of an equation (e.g., $4n + 9 = 7n$), the frequency of students using certain kinds of substitution increased significantly, compared with the cases in which the unknown appeared only once on one side of an equation.

Based on this finding, they claimed that the demarcation between arithmetic and algebra lies at a *cognitive gap*, which is characterized by students' inability to operate on or with the unknown, rather than at the *didactic cut* proposed by Filloy and Rojano (1984).

Mathematics Teachers' Understanding of Equation Solving

Relative to the magnitude of studies on students' understanding of equation solving, there are fewer and less systematic studies on teachers' conceptions. Below are a few that I have found.

Adi (1978) conducted a study on preservice elementary teachers learning the reversal method and formal method for solving equations. The result suggests a connection between the Piagetian developmental levels of these teachers and their performance on equation solving.

Attorps (2003) conducted a small-scale qualitative study with 10 secondary school mathematics teachers on their conceptions of equations. The results indicate that the teachers have a narrow understanding of the equation concept. This lack of insight relates to their own learning experiences, which mostly focused on acquiring procedural skills through mechanical drills.

When the function-based approach became popular in algebraic curricula and teaching, Chazan and his colleagues conducted a few studies on the status and changes in teacher knowledge. Chazan et al. (1999) interviewed one intern teacher who was employing graphing calculators to teach algebra but had never used graphing technologies in her own high school years. During the interview she was able to identify different uses of the Cartesian coordinate system in graphing, and she subsequently realized that her own instruction did not differentiate enough between solving an equation in one variable (e.g., $3x + 7 = 2(x + 5) + x - 1$) and solving a system of equations in two variables (e.g., $y = 3x + 7$ and $y = 2(x + 5) + x - 1$). Different conceptions of equations, equation solving, solutions, and the connections and contrasts among them become particularly relevant when teachers switch from equation-based algebra curricula to function-based, technology-intensive curricula and teaching.

With a similar goal, Yerushalmy, Leikin, and Chazan (2004) studied three mathematics teachers who were involved in an algebraic curricular change, from an equation-based to a function-based approach. As a result of teaching a different curriculum, the three teachers then had different strategies for solving equations than one would have expected they might have had before. The notion of an equation as a comparison of two functions (e.g., when we the values of the two functions $f(x) = 3x - 4$ and $g(x) = x + 2$, the result will be different when $x < 3$, $x = 3$, and $x > 3$. Solving the equation $3x - 4 = x + 2$ can then be viewed as one of those comparisons) helped all of them see particular equations differently from how researchers believed they had in the past. Two teachers were quite articulate about the ways in which the different views of equations and the different strategies for solving them were parts of different views of algebra. They switched between the two views in teaching and saw some benefits to a function-based approach, but had concerns about this approach's treatment of equations in two variables. The other teacher did not switch between the two views. He maintained a function-based perspective when dealing with equations in one variable but was not able to extend it into equations in two variables.

When the new approaches to school algebra curriculum are introduced by technological advances, they will pose potentially fascinating challenges to teachers and students, and also leave important questions for researchers to explore further (e.g., what sorts of knowledge teachers have of the school algebra curriculum and what sorts of knowledge are necessary for faithful implementation of particular approaches to the subject) (Yerushalmy & Chazan, 2002). Such questions are worth answering, particularly for those topics and goals, including solving linear and quadratic equations, common to both the conventional and new approaches.

EQUATION SOLVING IN SCHOOL ALGEBRA TEXTBOOKS

Equations and equation solving constitute a major theme in school algebra curricula. To examine which related topics are covered by current algebra curricula and which could be worthy of investigating in this study, I surveyed seven commercially published first-year algebra textbooks which vary in their structures and approaches:

- *Algebra 1*. CORD Communications (2004)
- *Algebra 1*. Holt, Rinehart, Winston (2007)
- *Algebra 1*. Prentice Hall (2007)
- *Algebra 1: Concepts and Skills.* McDougal Littell (2006)
- *Algebra 1: Integrations, Applications, Connections.* Glencoe (1998)
- *Discovering Algebra*. Key Curriculum Press (2007)
- *UCSMP Algebra*. Prentice Hall (2002)

Below, I discuss a few major topics that are common components of the textbooks, as well as concepts and processes in these topics that could potentially lead to assessment questions regarding the kinds of knowledge teachers have or use in teaching.

Linear Equations in One Variable

In most textbooks, the balancing method is the only symbolic method introduced for solving linear equations. It is the last method (using formal rules) in the list summarized by Kieran (1992) and reviewed earlier in this chapter (p. 45). One performs the same operation on both sides of an equation to isolate the variable and eventually figure out its numerical value. In the textbooks, balance scales or algebra tiles are often used first to illustrate visually some sample solving processes. The four operation properties of equality (addition, subtraction, multiplication, and division properties) are then formally introduced both as the foundation and as the content of the method. Finally, several equations are solved symbolically as examples and applications of the properties. Therefore, a typical curricular or learning trajectory goes through three conceptual levels: (a) manipulatives and visual demonstrations, (b) general principles, and (3) symbolic manipulations on specific equations.

Balance scales or algebra tiles are intuitive tools for representing general principles and processes. Effectively teaching with these tools may go beyond familiarity and require teachers' sensitivities about the limitations of these manipulatives, the similarities and differences between them, and whether learners could smoothly transition from using manipulatives on special cases to solving general types of equations completely symbolically. For instance, an equation like $3x + 5 = 10$ cannot be completely solved with either manipulative since the solution is a decimal number. Whether an equation such as $3x + 10 = 4$ can be solved with a balance scale is arguable. Two of the books actually use negative weights in pictures of scales. So, the question we can ask is, what could be the psychological effects of using negative weights which do not correspond to signified real-world objects? Similar questions can be asked for representing negative terms with algebra tiles, of which sense can only be made by the notion of algebraic sum.

Tthe four properties of equality coexist with a particular constraint: We can be sure that the equations $f(x) \pm \mathrm{A} = g(x) \pm \mathrm{A}$, $\mathrm{A}\cdot f(x) = \mathrm{A}\cdot g(x)$, and $f(x)/\mathrm{A} = g(x)/\mathrm{A}$ ($\mathrm{A} \neq 0$) are equivalent to $f(x) = g(x)$ only when A is a number or a numerical expression. When A is a variable expression, the solution set may not be preserved (e.g., $2x = -6$ is not equivalent to $2x + \log x = -6 + \log x$). Further, a transformation on an equation may lead to an extraneous or lost root. For example, squaring on both sides of the equation $x = -1$ would introduce an extraneous root, $x = 1$. In general, we have the following theorem:

> Let h be a one-one function. Then for all x in the domains of f and g for which $f(\mathrm{x})$ and $g(\mathrm{x})$ are in the domain of h, $f(x) = g(x)$ if and only if $h(f(x)) = h(g(x))$. (Usiskin, Peressini, Marchisotto, & Stanley, 2003, p. 164)

It would be important to find out whether secondary school mathematics teachers are aware of the constraint on the four properties and able to provide instances of transformations that do not preserve solutions because the balancing method is often simplistically phrased as "doing the same thing on both sides". Similarly, there is the so-called "The Golden Rule of Algebra" which claims that "Whatever you do on one side of an equation, you do the same on the other".

One textbook introduces and discusses in detail an alternative method: the undoing method (or working backward), which is also included in Kieran's (1992) list (p. 45 of this chapter). A more detailed process for solving an equation with the undoing method follows. Take the equation $2x + 5 = 11$ as an example, it can be viewed as a sequence of two operations on x (multiply by 2 and add 5):

$$x \xrightarrow{\times 2} 2x \xrightarrow{+5} 11$$

To find the value of x, we begin with 11, undo the above two operations in reverse order, and get $x = 3$ at the end:

$$3 \xleftarrow{\div 2} 6 \xleftarrow{-5} 11$$

The method is based on a procedural (rather than structural) view of equations as well as on the idea of inverse operations. But unlike the use of inverse operations in the balancing method in which the inverses are applied on both sides and the order does not matter much, the undoing method can only be directly applied to equations with the unknown appearing only once, and it has to follow a unique path from the very end result backward, step-by-step to the unknown.

The concept of equivalent equations is defined in most textbooks, mainly during the introduction of the balancing method. For some reason, the topic is never brought up again in later sections and chapters. Equivalence is a fundamental concept because it characterizes the relationship between an equation and another one generated through certain transformations within one side or on both sides of the equal sign. As previously discussed regarding the balancing method, it also explains when and why a solving process may or may not preserve the solution set.

Meanwhile, although the definition of equivalent equations refers to sameness in solution sets, one does not always have to solve the equations in order to determine equivalence. When two equations are connected by a solution-preserving transformation, their equivalence is guaranteed. In that sense, the concept of equivalence adds weight to algebraic transformations in the studying of equation solving.

Linear Equations in Two Variables

One overarching issue here is the use of the term *equation*. Most textbooks have several chapters with the term "linear equations" in their titles, but the contexts are different: a linear equation to be solved is typically in one variable, while a linear equation to be graphed or that describes a line is typically in two variables. The dimensions of the solutions are different. Nonetheless, there are still connections between these two situations. For example, we can not only solve the linear equation (in one variable) $3x + 5 = 11$, but also graph it on the real number line, which gives the point $x = 2$. For a linear equation in two variables, such as $y = 3x + 5$, we can not only graph the line it represents but also solve it (the solution set is all number pairs $(x, 3x + 5)$, where x is any real number). Further, to solve the equation $3x + 5 = 11$ is equivalent to figuring out the input to the function $f(x) = 3x + 5$ when the output is 11. Such a view links the solving and graphing activities by the concept of functions and relations.

It is desirable for algebra teachers and students to have not only a unified concept of (linear) equations, but also to understand the subtle differences in various contexts or perspectives and the connections among them. Equations to be solved, equations of curves, and equations of functions, together with equations that fit data, constitute four major topics in modern school algebra. They are overlapping but distinct because they are fundamental to four different mathematical subjects: classic algebra (theory of equation solving), analytic geometry (the algebraization of geometric curves), calculus (which places functions as a core object for study), and numerical analysis (finding equations that best simulate given data sets).

System of Linear Equations

There is a connection between solving a system of linear equations, $\begin{cases} y = ax + b \\ y = ca + d \end{cases}$, and solving a related linear equation, $ax + b = cx + d$: in both cases, we can graph the two functions $y = ax + b$ and $y = cx + d$ and then examine the intersection. A fundamental difference, though, lies in the dimensions of their solutions: If the solution to the system is the ordered pair (s, t) (i.e., the intersection point of the two function graphs in the x-y coordinate plane), then the solution to the equation is the number s (i.e., a point on the x-axis). It is basic knowledge for algebra teachers to understand the similarities and differences between the graphical solutions in these two cases.

Quadratic Equations

Two common methods for solving quadratic equations are introduced in all textbooks: the factoring method and the quadratic formula.

Several textbooks portray algebra tiles to illustrate the factoring process. On the one hand, algebra tiles can only help to figure out binomial factors with integer coefficients. If the tiles that represent a trinomial cannot be arranged into a rectangular shape, the trinomial cannot be factored. And this aligns with one of the facts associated with the symbolic factoring process: since the typical trinomial provided in the form x^2 +

$by + c$ has integer coefficients, it is true that the two roots must be factors of the constant c. In other words, the "factorability" of a trinomial is discussed in the integer set most of the time. However, such an implicit assumption does not have to be true.

On the other hand, different textbooks use algebra tiles in distinct ways: one book only gives examples of trinomials with positive coefficients, yet another book shows trinomials with combinations of positive and negative coefficients (through assigning signs to the tiles). When "negative" tiles are involved, the interpretation of the "total area" of the rectangular tiles becomes "the algebraic sum" of the smaller tiles, and this could be confusing to some students. For example, when being factorized, the two distinct trinomials $x^2 + 4x + 4$ and $x^2 - 4x + 4$ have exactly the same size and shape in the algebra tile layout. Perhaps this is not something easy to understand for algebra students or easy to teach for algebra teachers.

The quadratic formula is the most general method for solving quadratic equations and is derived from another general method: completing the square. However, in many algebra textbooks, the latter method is either introduced after the quadratic formula or simply not introduced at all. After all, regenerating the method is an intensive symbolic reasoning process and could be overwhelming to many learners and even some teachers. One question that could be asked is whether students can learn the quadratic formula equally well with or without first being taught the completing the square method.

Rational and Radical Equations

A key issue for these types of equations is the possibility of having extraneous roots or lost roots. The textbooks introduce these concepts and teach how to avoid having such roots through checking answers, but it is not clear if teachers and students are knowledgeable about and sensitive to the typical kinds of transformations on rational and radical equations that may likely evoke extraneous or lost roots (e.g., multiplying both sides by a binomial or squaring both sides).

IMPLICATIONS FOR FURTHER RESEARCH

The five perspectives reviewed early in this chapter decompose (mathematical) knowledge for teaching into various categories. Despite the differences in scope and structure, these perspectives implicitly address three common components: teacher knowledge of (a) the mathematical subject matter, (b) how students learn and understand mathematics, and (c) how to present mathematical ideas into appropriate forms that learners could comprehend. It is reasonable to expect that these three aspects of teacher knowledge be integrated or reflected, in one way or another, in any theoretical frameworks that attempt to characterize the knowledge demand for teaching mathematics.

The review of mathematical procedures, algorithms, and procedural and conceptual understanding provides insights and raises issues for future studies on the teaching and learning of procedures and, in particular, for my proposed study of teachers' understanding of equation solving for teaching. On the one hand, in teaching standard algorithms and alternative strategies for mathematical procedures such as equation solving, teachers' conceptions of the procedures' features, roles, and interrelationships would be crucial to what and how they teach. On the other hand, the fundamental relationship between procedural and conceptual understanding still deserves attention nowadays. Examining teachers' conceptions of such relationship, specific to equation solving procedures, would help us to understand better their teaching practices.

There is a surprising contrast between the studies on students' understanding of equations and equation solving and those on teachers' understanding, in terms of quantity and coverage of topics. The few studies I have found on teachers mostly center around their subject matter understanding; therefore, future studies on teachers' knowledge of mathematics for teaching equation solving need to expand to cover teacher knowledge of learner conceptions and of pedagogical presentations.

The infusion of instructional technology in school classrooms puts forward new approaches to teaching and learning algebra, while the conventional approaches coexist. This makes the implementation of the already diverse algebra curricula even more complex. Although equation solving is a fundamental topic commonly addressed by different curricula and different approaches, there are not sufficient research findings to inform us about how algebra teachers teach equation solving and what kinds of knowledge and reasoning are underlying. Nonetheless, both the review of previous studies on the typical strategies used by students and the survey of various first-year algebra textbooks produced a collection of topics and issues that my research design and instrumentation could address.

Chapter 3 Conceptual Framework

ASSUMPTIONS ABOUT TEACHER KNOWLEDGE

A major assumption I have about teacher knowledge and teacher learning is that there exists a knowledge base for teaching a certain school subject, which includes, but is not limited to, knowledge of (a) the fundamental concepts, processes, and ways of thinking and reasoning about the subject, and their connections to those in prerequisite and more advanced subjects; (b) how student textbooks and other materials typically present and sequence the major topics in the subject; (c) some general or specific ways of introducing, representing, and explaining a particular topic to learners, responding to their ideas and questions, and orchestrating and facilitating interactions among learners; and (d) learners' common conceptions, mistakes, and difficulties in learning a particular topic, as well as their current status in cognitive development.

Such a knowledge base has three major characteristics:

1. It is developing and changing, rather than fixed. As described above, it is the minimal set of expertise that teachers develop through preservice preparation, inservice professional development, as well as their teaching practices. The longer a teacher has taught a subject, the richer and more stable his or her knowledge base tends to become. Nonetheless, the knowledge base alone still may not necessarily be sufficient for a teacher to handle all topics in the subject, students with various characteristics and needs, and different classroom environments. Actually one of the challenges for inservice

teacher professional development programs has been some teachers' reluctance to change, including updating their knowledge base for teaching.

2. It is a core body of knowledge that teachers may need to modify, reassemble and adapt before applying in various teaching and learning situations. In dealing with a particular teaching task, the parts of the knowledge base upon which a teacher would draw upon and how these components are used depend on factors such as the structure and depth of the knowledge base, the teacher's beliefs about and preferences in teaching and learning, and the school and classroom context. Therefore, the application of a knowledge base is teacher-dependent and context-sensitive.

3. It is often implicit. Not only is teacher knowledge invisible to others and can be accessed mostly indirectly via methods such as observation of teachers' actions and analysis of teachers' responses to written or oral questions, it may not be in the teachers' metacognitive system, either (i.e., teachers may not be fully aware of what knowledge upon which they have drawn in making a certain conclusion or decision). One of the goals of teacher preparation and professional development would be to make such knowledge more explicit, through teachers' reflections and discussions on their practices and decision-making processes.

With the above assumptions, I am developing a conceptual framework that could ideally characterize the structure of the mathematical knowledge base for teaching equation solving and other mathematical procedures.

THE PROVISIONAL FRAMEWORK

A conceptual framework is a critical element of scholarly inquiry and serves multiple purposes for both individual researchers and the entire field (Mewborn, 2005). Specific to this dissertation study, the conceptual framework I developed is a model that outlines the structure and components of the notion in focus (mathematical knowledge for teaching equation solving). The framework plays the role of a lens through which the research questions could be dissected and rephrased, the selection of methods and the design of instruments could be well structured and aligned with the research questions, and the data analysis and sense-making could be focused and grounded.

To increase the construct validity of this study, I followed a few basic strategies suggested by literature on research methods (Garson, 2005; Trochim, 2001). I developed my framework on the basis of existing research findings and characterizations of related notions. Basic components of the notion in focus were reflected in the framework, and in this chapter I will provide operational definitions for the components and concepts involved. Later I will discuss how the measurements were designed to strengthen further the construct validity (Chapter 4) and how consistent the data are with the conceptualizations in the framework.

The five perspectives reviewed in the previous chapter have different scopes but all with a certain level of generalizability. Shulman’s theory on content knowledge for teaching is the most general one and could be applied to all school disciplines. Leinhardt and Smith’s perspective on subject matter knowledge underlying expertise in teaching, as well as the scheme proposed by Ball, Bass, and their colleagues, is meant to cover all school mathematics. The two frameworks developed by Artigue and colleagues and Ferrini-Mundy and colleagues are about knowledge for teaching a fundamental subject in school mathematics, algebra, but all or part of those two constructs have the potential to be modified and generalized to other mathematics subjects.

The conceptual framework I developed for this study not only reflects the characteristics of the focus topic, algebraic equation solving, but also applies to a variety of other mathematics subjects of the same nature. The first two groups of literature reviewed in the previous chapter (i.e., (a) the five perspectives on content knowledge for teaching, and (b) studies on mathematical procedures, algorithms, and procedural knowledge) revealed that such a framework should integrate at least the following two generalizable dimensions:

1. One dimension identifies the basic components of mathematical knowledge for teaching. A few common domains of knowledge were observed from the five reviewed perspectives: knowledge of the subject matter (mathematics as a body of concepts, methods, theorems, and reasoning techniques), knowledge of learner conceptions (mathematics as conceptualized by the learner), and knowledge of didactic representations (mathematics as presented by instructional media and materials). This triad of knowledge is not necessarily limited to mathematics but has great potential to be adopted and integrated into a framework for other disciplines.

2. A second dimension characterizes mathematical procedures, which include equation solving and a variety of other operations and processes, that permeate mathematics studies at various levels. The literature review brought to my attention some

central issues related to the teaching and learning of mathematical procedures, especially the differences and connections among various algorithms and strategies for a certain procedure, and the relationships between a procedure and related concepts. At the generic level, a mathematical procedure involves three conceptual aspects: (a) basic (or "standard") algorithms, (b) alternative algorithms and strategies, and (c) related concepts, procedures, and properties.

The above conceptualizations scaffolded the provisional conceptual framework that I built to characterize teachers' mathematical knowledge for teaching procedures (Figure 3.1).

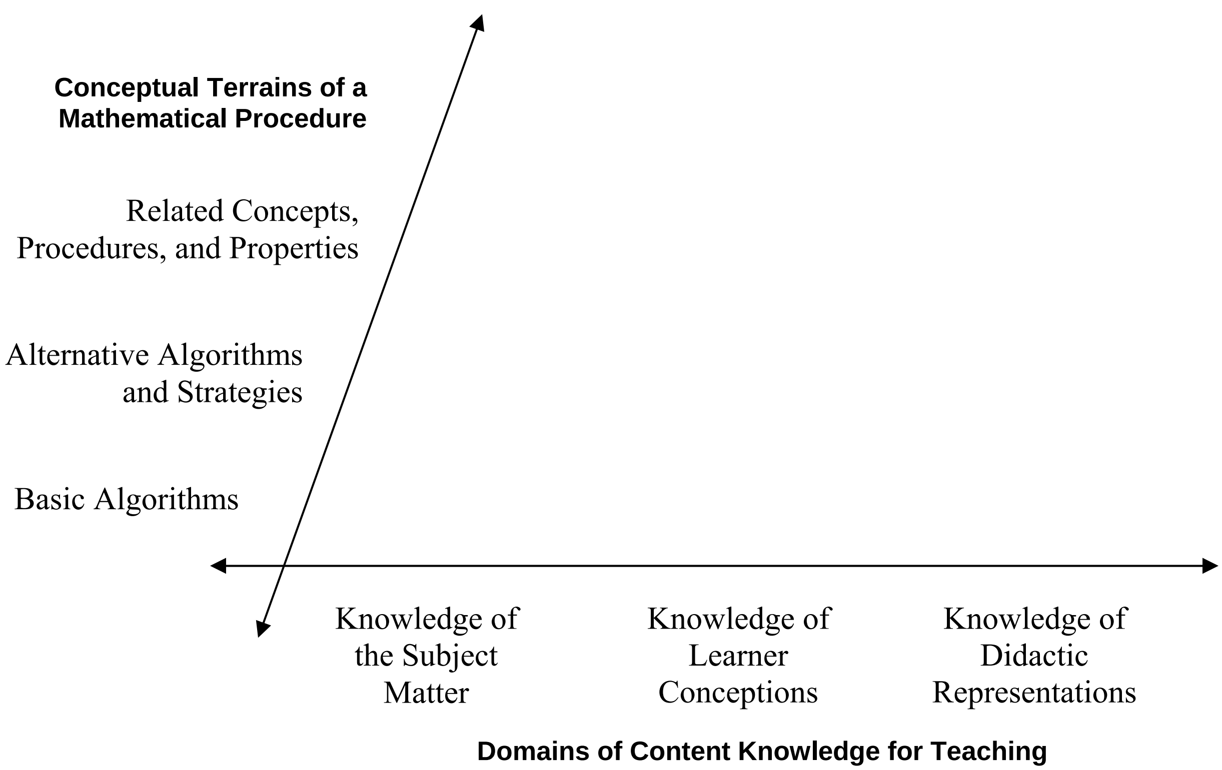


Figure 3.1 The provisional framework

The framework encompasses two dimensions: (a) domains of content knowledge for teaching and (b) conceptual terrains of a mathematical procedure. Both of them include three subcategories. The direction of an axis or the position of a category on an axis does not indicate the level or priority of the category. In other words, the three subcategories in each dimension are neither independent nor hierarchical.

DOMAINS OF CONTENT KNOWLEDGE FOR TEACHING

The three categories in this first dimension correspond to three forms of content knowledge for a particular academic discipline. In defining each of the categories, I will use mathematics as the particular example.

Knowledge of the Subject Matter

This domain is teachers' knowledge of the content in its subject matter form, both as an academic discipline and as a course of study. The subject matter is typically an organized system consisting of interrelated concepts, properties, and methods. Below is an excerpt about mathematics from the popular online dictionary wikipedia.com:

> Mathematics is the body of knowledge centered on concepts such as quantity, structure, space, and change, and also the academic discipline that studies them…Through the use of abstraction and logical reasoning, mathematics evolved from counting, calculation, measurement, and the systematic study of the shapes and motions of physical objects. Mathematicians explore such concepts, aiming to formulate new conjectures and establish their truth by rigorous deduction from appropriately chosen axioms and definitions. (http://en.wikipedia.org/wiki/Mathematics)

The description reveals some fundamental elements of mathematical subject matter: concepts to be studied (quantity, structure, space, and change), thinking and reasoning methods (abstraction, inductive and deductive reasoning, etc.), mathematical activities (counting, calculation, measurement, conjecturing, proof, etc.) and their products (definitions, axioms, theorems, etc.).

In both research and education, the subject matter of mathematics is organized into major branches such as arithmetic, algebra, number theory, discrete mathematics, analysis, geometry, trigonometry, topology, applied mathematics, and probability and statistics. Each of these branches can further be divided into a variety of subjects. For example, the branch of algebra includes three major sub-branches: elementary algebra, abstract algebra, and linear algebra. And abstract algebra itself has specialized theories in groups, rings, fields, and other fundamental structures. The two most influential professional publication reviewing databases in mathematics, *Mathematical Reviews* (MR) and *Zentralblatt MATH* (Zbl), have developed and used the Mathematics Subject Classification (MSC) system to code publications. The current version of the system (MSC2000) classified 65 primary mathematical branches and a total of more than 5000 secondary subjects.

The subject matter of mathematics does not exist in an abstract or transcendental form. Instead, it appears in journals, lecture notes, monographs, books, textbooks, and others types of publications in the forms of definitions and properties of concepts and other mathematical objects, algorithms and strategies for operations and procedures, symbol and representation systems, reasoning methods and problem solving strategies, etc.

Teachers' subject matter knowledge is divided into categories by the perspectives reviewed in Chapter 2: for instance, Shulman's (1986a) definitions of substantive and syntactic structures, Ball, Bass, et al.'s (2005, 2006) distinction between common content knowledge and teachers' specialized content knowledge, and Ferrini-Mundy et al.'s (2003, 2004) division between school mathematics knowledge and advanced mathematics knowledge.

A few previous studies have demonstrated positive connections between teachers' subject matter preparation and student achievement, but there was also evidence of a "threshold effect" on student achievement, which refers to the minimal additional effect of teachers' mathematics studies beyond five undergraduate courses (Wilson & Floden, 2002; Wilson et al., 2001). In other words, a strong mathematics background is a necessary but not a sufficient condition for effective mathematics teaching. What makes effective mathematics teachers may also depend on their knowledge of other forms of mathematics: the mathematics that is comprehended by learners, and the mathematics that is pedagogically represented.

Knowledge of Learner Conceptions

This domain represents teachers' knowledge of the cognitive form of the content, in learners' mental representations. Such content results from the learners' interactions with the teacher, the subject matter, other learners, instructional technologies, and other factors in the learning environment. Such knowledge could be partially observed from and assessed by the questions that learners ask, and their responses given to questions raised by the teacher or other learners in classroom interactions and those given in exercises, homework, and exams. To teach effectively, teachers need to be interested in and very knowledgeable about the level and depth of learners' prior and current understanding; the nature of their pre-conceptions, misconceptions, mistakes, errors, difficulties, unconventional or non-typical ideas and ways of thinking; as well as their learning trajectories across topics, sections, chapters, units, subjects, and even the entire school curriculum.

I consider the cognitive form of mathematics as part of a broader notion of mathematics, rather than as a separate, cognitive entity. Mathematics, as a body of knowledge, is not a system of absolute truths that are free of fallacies. During the historical progress of mathematics, human knowledge about mathematics has developed in an upward spiral. Undefined or ill-defined concepts, vague definitions, incomplete or incorrect beliefs or statements, unproved hypotheses and conjectures: these have appeared constantly in the mainstream of mathematics development. Learners' conceptions, misconceptions, and learning trajectories often mirror the historical paths through which mathematics has gone. Those premature or underdeveloped forms of mathematics, either in history or in learners' understanding, are non-negligible components of the ever-evolving system of mathematics knowledge.

The five perspectives reviewed in Chapter 2 have identified representative knowledge of learner conceptions, such as "the understanding of classes of student errors" (Leinhardt & Smith, 1985, p. 247), "an understanding of what makes specific topics easy or difficult for a certain group of learners" as part of pedagogical content knowledge (Shulman, 1986a, p. 9), or categories such as knowledge of content and students (Ball, Bass, et al., 2005, 2006) and the cognitive dimension in Artigue et al. (2001).

One important model of mathematics teacher preparation and professional development programs in the United States focuses on increasing teachers' knowledge of student learning. One of the most successful and influential programs is the Cognitive Guided Instruction (CGI) project, which has targeted elementary level mathematics concepts (Carpenter, Fennema, & Franke, 1996; Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Wilson & Berne, 1999). Studying teachers' knowledge of how students understand secondary school mathematics topics could lay solid foundation for professional development programs that are similar to CGI in design but that focus primarily on secondary school mathematics.

Knowledge of Didactic Representations

This domain has to do with teachers' knowledge of the pedagogical form of the content as represented by instructional media (e.g., textbooks, manipulatives, visual aids, and electronic technologies) and presented by various teaching strategies. Such content is the intermediate product when the subject matter is being unpacked, linked, reorganized, and tailored by the teachers for specific groups of learners through (a) particular ways of sequencing content units and topics, and (b) the uses of all sorts of examples, metaphors, models, questions, explanations, tasks, tools, technologies, etc.

As is the case in any other discipline, knowledge and theories of mathematics may exist as systems of perceptions, conceptions, propositions, beliefs, and schemas that inhibit in an individual's mind or that are shared by a community of learners, users, or practitioners. Whenever an individual has the intention to communicate his or her own mathematical understanding with others, he or she has to rely on certain media, such as spoken languages, writings (with words, symbols, or notations), drawings (pictures, graphs, charts, diagrams, tables, etc.), and gestures. The outcome of a communication process would be heavily dependent on the selection, use, and interpretation of the media by each participant.

Mathematics, as well as its teaching and learning, is a special form of communication between teachers and students, and among the students themselves. At the macro level, mathematics curriculum standards and textbooks specify the coverage and sequencing of topics for a certain grade level or course of study. This influences the organization of mathematics content presented to the students. At the micro level, the definitions, examples, explanations, and exercises given in textbooks; the examples, tasks, exercises, metaphors, explanations, and manipulatives that teachers use; the questions teachers ask; and the ways they respond to students' questions and thoughts are all factors that influence the relationships between the mathematics content and the students. At both levels, mathematics teachers' understanding and use of the materials, media, artifacts, and strategies play important roles in shaping the nature, structure, and quality of the mathematics subject matter that the students experience and, ultimately, the conceptions that the students acquire (i.e., the cognitive form of mathematics).

The previously reviewed five perspectives also provide categories of knowledge of didactic representations, such as lesson structure knowledge (Leinhardt & Smith, 1985), knowledge of content and teaching (Ball et al., 2005, 2006), and the didactic dimension in Artigue et al. (2001). In Shulman (1986a) and Ferrini-Mundy et al. (2004), it is part of the pedagogical content knowledge category and teaching knowledge category, respectively. Shulman's "curricular knowledge" could also be viewed as one special type of knowledge of didactic representation.

Mathematics teachers may acquire knowledge of didactic representations through their school education, mathematics teacher preparation and professional development, and teaching practices. They need to learn how to teach mathematics not only as general pedagogical strategy, but also as content-specific pedagogical reasoning. One consequence of the lack of the latter type of knowledge would be that some teachers make pedagogical decisions without conducting thorough analysis of the mathematics subject matter. In such a case, treatment of the subject matter by the teacher and students could be very superficial, even if the pedagogical effect could be very engaging and motivating.

Below, I use an example to illustrate the three domains of content knowledge for teaching. The topic I chose is at the center of this study: linear equation solving.

An Example of the Three Domains of Knowledge

In teaching how to solve linear equations (in one variable), mathematics teachers' subject matter knowledge would at least include understanding of the concepts of equations and solutions, inverse operations, the four properties of equality, and various solving strategies (trial-and-error, the balancing method, the undoing method, using graphing calculators to trace the line, examining the numerical tables, and finding the intersection of the two lines as represented by the expressions on the two sides of the equation). A more in-depth understanding would include knowing the connections among these concepts and methods, what types of equations to which each method would apply, and the strengths and limitations of each method, etc.

As was summarized in Chapter 2, students make various mistakes and experience various difficulties in learning and using the concepts and methods for solving linear equations. Some of the underdeveloped conceptions (such as viewing an equation as a command or an action that leads to a certain result) have roots in elementary school mathematics studies. Teachers' thorough understanding of student conceptions in equation solving would involve not only recognizing the mistakes or difficulties when they are present, but also reasonably concluding the most likely causes and the common nature of similar mistakes or difficulties.

Teachers' knowledge of didactic representations regarding linear equation solving has at least two levels. At the curriculum level, it involves knowledge of how to sequence linear equations from the simplest to the most complicated types (one-step, two-step, multi-step, etc.), and the mathematical and cognitive reasoning behind such way of sequencing. At the topic level, it is important to use hands-on and visual tools (such as balance scales, algebra tiles, pictures, or function graphs with computers or calculators) to represent the equations, the solving process, and the final results. It is crucial for teachers to go beyond being able to solve a given equation with one of these representations and develop sensitivity to the strengths and limitations of each approach.

CONCEPTUAL TERRAINS OF A MATHEMATICAL PROCEDURE

This dimension consists of three sets of fundamental entities and utilities that a mathematical procedure involves: basic algorithms, alternative algorithms and strategies, and related concepts and procedures. I will first provide working definitions for the key terms and a concise characterization for each entity, then further clarify all three entities by giving two specific examples.

Definitions

As mentioned in Chapter 2 (p. 33), my definition of a *mathematical procedure* is an operation or process that acts on initial inputs and produces desired results. These inputs and outputs could be verbal, numerical, symbolic, graphical, or geometrical. In school mathematics, typical and fundamental procedures include counting; performing the four basic operations on whole numbers, integers, fractions, and other forms of rational numbers and real numbers; simplifying fractions; converting among fractions, decimals, and percents; transforming expressions and equations; graphing functions and equations; finding intercepts; etc. College-level mathematics courses may involve more advanced procedures, such as the long division of polynomials, computing the limits of sequences or functions, finding the derivatives and integrals of given functions, and finding solutions to a different equation.

A *strategy* is sequence of actions and reasoning for carrying out a procedure, which may or may not be well-defined or generalizable to a variety of instances. Quite often, different strategies may exist for carrying out a particular procedure. For example, one may be able to solve a quadratic equation by guessing and checking, using the factoring method, completing the square, applying the quadratic formula, or examining the table or graph of the corresponding parabola. The strategies could have been established by intellectuals across mathematics history and in different cultures, or invented by individual users and learners.

If a general strategy is clearly routinized, widely adopted, and able to produce accurate results, it is often named an *algorithm*, or a *rule*, or a *formula*. In other words, an algorithm is a well-defined, finite, and generalizable sequence of actions and reasons for carrying out a mathematical procedure and yielding definite results.

Basic Algorithms

These are the general and precise algorithms for carrying out the procedure, and are typically or conventionally taught as the main methods in most school mathematics curricula. Generally and relatively speaking, such algorithms are also efficient for a variety of cases, if not for all.

Alternative Algorithms and Strategies

Among all possible strategies for carrying a procedure, some are less often formally introduced in school mathematics curricula. These are the alternative algorithms and strategies. They may have lower levels of generality, efficiency, or precision than the basic algorithms.

These alternatives may have been used historically by different civilizations in different regions or constantly “invented” by learners in mathematics classrooms.

Oftentimes, the basic algorithms may be less transparent than the alternative strategies, which can be viewed as the price paid for higher levels of generality or efficiency.

Related Concepts, Procedures, and Properties

These are the concepts, procedures, and properties that serve as direct foundations for the given procedure or build directly upon the given procedure.

Below, I use two examples to illustrate the three conceptual aspects of a mathematical procedure. One example is a key procedure in elementary school mathematics curricula: multi-digit whole number subtraction. The other example is solving linear equations in one variable.

Two Examples of the Conceptual Terrains

1. Multi-digit whole number subtraction

Suppose we are trying to compute 73 – 48. The basic (or standard) algorithm for subtraction is to write the minuend (73) on top of the subtrahend (48) and align them by places, then to begin the subtraction process from the lowest (ones') place and move toward higher places. On each place, if the number on the top is smaller than the one on the bottom, we need to move over to the next higher place to "borrow 1." In this case, when we borrow 1 from 7, we get 13 – 8 = 5 on the ones' place. And the tens' place becomes 6 – 4 = 2. Therefore, the result is 25.

There are several alternative strategies for subtraction. For example:

- Partial Differences: Aligning the two numbers, doing subtractions on each place, generating negative differences without borrowing if the number on the top is less than the number on the bottom. In this case, 3 – 8 = -5 and 7 – 4 = 3, so the difference is 30 + (-5) = 25.

- Adding Up: Beginning with the smaller number and adding “chunks” of numbers until reaching the larger one. The difference will be the sum of the chunks. In this case, 48 + 2 = 50; 50 + 20 = 70; and 70 + 3 = 73; therefore, the difference is 2 + 20 + 3 = 25.

- Compensation: Changing one number to its closest multiples of 10 by adding or subtracting a small quantity, then compensating that quantity after the subtraction. To solve 73 – 48, we first solve 73 – 50 = 23. Because 50 is 2 more than 48, we need to add 2 to 23 and get 25.

The standard algorithm can be used with any multiple-digit subtraction, and overall, it is the most efficient among all strategies in the sense that we can keep moving from the lower to the higher places and the differences will all appear in one row. The three alternative strategies may be easier to make sense of (i.e., more transparent) than the standard algorithm, which is quite “packed,” but they are not as efficient as the standard one (e.g., there are more steps involved in the Partial Difference method and more mental estimations involved in the Adding Up method). The Compensation method has lower generality because it makes less sense when the subtrahend is not close to a multiple of 10 (e.g., 73 - 45).

The multi-digit subtraction procedure is based on base-10 and place value concepts, as well as on single-digit subtraction procedures. It is closely tied to the addition procedure in at least two ways: (a) they are inverse operations (i.e., if $c - b = a$, then $c = a + b$), and (b) subtracting a number is equivalent to adding its opposite (i.e., $c - b = c + (-b)$). An immediate application of subtraction procedures can be found in a multi-digit division procedure, also, in two ways: (a) division can be carried out through repeated subtractions, and (b) the long division algorithm does involve subtraction as middle steps.

2. Solving linear equations in one variable

Several methods for solving linear equations have been identified by Kieran (1992) and reviewed in Chapter 2. The standard algorithm is the *balancing method*, that is, performing the same inverse operations on both sides of the equation (method No. 7 in Kieran's list). It works for all equations in the general form $ax + b = cx + d$.

For those additive inverse operations involved in a balancing process (i.e., additions and subtractions), the *transposition method* (moving a term to the other side of the equation and changing its sign, method No. 6 in Kieran's list) could be used as a shortcut. It is more efficient than the balancing method but less transparent or general; we would have to use a different set of rules for applying multiplicative inverses (e.g., moving the coefficient of a term from one to the other side of the equation by dividing it into each term on the other side).

The *undoing method* is also introduced in some algebra textbooks (method No. 4 in Kieran's list). Instead of viewing an equation as an equivalence relation between two expressions or quantities, the undoing method treats an equation as a sequence of operations that links the unknown to an end result, which is closer to the way that equalities are viewed in elementary school arithmetic. Hence, it may make more sense to some learners and users than the balancing method. A major limitation of this method is that it only applies to $ax + b = c$ types of linear equations.

With modern technology, at least three other methods can be used to solve linear equations:

- Tracing a line: For instance, to solve $3x + 5 = 7$, we could graph and trace the line $y = 3x + 5$ to figure out for which value of x the value of y would be 7.
- Checking a numerical table: For an equation such as $3x + 5 = 4x – 2$, generate a table of values for the two expressions on the two sides and find out which value of x would make the values of the expressions equal.
- Intersecting the lines: Graph the two lines $y = 3x + 5$ and $y = 3x – 2$. The x-coordinate of the intersection would be the solution.

The strength of these three methods is that they involve graphs or numerical tables, hence may be more “intuitive” or “visual” than the symbolic approaches. Conversely, they may not always yield accurate results (e.g., when the solutions are non-integers), and the settings of the tables’ coordinate systems could become barriers when the answers fall far outside the range and cannot be easily located.

Solving equations involves the basic concepts of equalities, equations, solutions, and equivalent equations. The balancing method is based on the four basic properties of equality, and utilizes the concept of inverse operations and the procedures of performing inverse operations. The study of equations and equation solving is directly followed by the studies of inequalities, systems of equations, and systems of inequalities.

SUMMARY

This chapter establishes a provisional conceptual framework that characterizes the focused concept of the study: teachers' mathematical knowledge for teaching equation solving as two-dimensional. On the one hand, mathematical knowledge for teaching has three basic domains: knowledge of the subject matter, knowledge of learner conceptions, and knowledge of didactic representations. On the other hand, equation solving is a mathematical procedure that involves three conceptual terrains: basic algorithms; alternative algorithms and strategies; and related concepts, procedures, and properties. These characterizations provide guidance for the research instrument design and data analysis that will be discussed in the next two chapters.

Chapter 4 Research Methods

OVERVIEW

On the basis of the reviews and discussions in the previous two chapters, this chapter first rephrases the two main research questions with more particularities. The rest of this chapter addresses the major design issues of the study: the population, sampling frame and methods, the participants, research instruments, data collection procedures, and data analysis strategies.

SPECIFIED RESEARCH QUESTIONS

Based on the literature review in Chapter 2 and the conceptual framework established in Chapter 3, teachers' mathematical knowledge for teaching can be perceived as the integration of three interconnected domains: knowledge of the subject matter, knowledge of learner conceptions, and knowledge of didactic representations. Meanwhile, a mathematical procedure is tied to three fundamental conceptual entities: (a) basic algorithms; (b) alternative algorithms and strategies; and (c) related concepts, procedures, and properties. To analyze one's conception of a mathematical procedure, we could examine his or her understanding of the nature and roles of each individual entity, the interrelationships, and the nature and roles of mathematical procedures in general.

As a result of these conceptualizations, research question (I), what kinds of mathematical knowledge would secondary school algebra teachers draw upon when pondering upon problem situations that could arise in the teaching and learning of algebraic equation solving, can be broken down to three more specific questions:

(I.1) When pondering problem situations that could arise in the teaching and learning of equation solving, what would algebra teachers draw upon within and across the three basic domains of knowledge: (a) knowledge of the subject matter, (b) knowledge of learners' conceptions of the mathematics, and (c) knowledge of didactic representations of the mathematics, and how are these knowledge domains used?

(I.2) How would algebra teachers understand the features and roles of the basic and alternative algorithms and strategies for solving a particular type of equation, and what are their expectations in regard to teaching multiple strategies?

(I.3) How would algebra teachers understand the role of mathematical procedures and routines in general?

Basic variables in teachers' academic backgrounds and teaching experiences that are relevant to this study include college major, course-taking in advanced mathematics and mathematics education, the type of algebra courses they have been teaching, number of years of teaching algebra, etc. Therefore, research question (II) (How is the mathematical knowledge upon which these teachers draw related to their academic backgrounds and algebra teaching experiences) can be rephrased into the following:

(II.1) How may the mathematical knowledge upon which algebra teachers draw for teaching equation solving be differentiated by teachers' basic characteristics, such as

(a) college major, (b) course-taking in advanced mathematics and mathematics education, (c) school algebra course-teaching, and (d) number of years of teaching algebra?

(II.2) What other factors in teachers' backgrounds and experiences may have a major impact on teachers' knowledge for teaching equation solving?

In an attempt to address both the breadth and depth of the above questions, while also taking into account the constraints in time, resources, and accessibility to schools and teacher participants, I decided to conduct the proposed study with a relatively small sample of secondary school algebra teachers who have varied backgrounds and experiences but all hail from the same state in the US. Participant recruitment and data collection involved two phases: (a) administering the written instruments (academic background questionnaire and written-response assessment) and (b) conducting semi-structured follow-up interviews. A mix of quantitative and qualitative methods was employed in the data analysis.

POPULATION, SAMPLING FRAME, AND PROCEDURE

Population

The population for this study is secondary school (sixth to twelfth grade) mathematics teachers in the state of Texas who have taught first-year algebra in the last three years or are currently teaching such a course.

Sampling Frame

Because research question (II) is intended to examine whether the variance in teachers' knowledge and reasoning can be attributed to variables in teachers' academic backgrounds and teaching experiences, the selection of the population, as well as the sampling and participant-recruitment procedures, has to guarantee diversity in teachers' backgrounds and experiences. In this process, I have mainly considered three key parameters:

1. Policy and curricular contexts

As pointed out in Chapter 1, school curriculum standards in the US vary considerably across states, in terms of content expectations, focus, scope, and sequencing, even for the same school mathematics subjects. Therefore, a manageable and reasonable population for this dissertation research would be algebra teachers from a single state which has a set of long and well established curriculum and assessment standards for secondary school mathematics and, in particular, algebra. On the one hand, this could control the variable of state-level policy context. On the other hand, this would allow for enough variability in the mathematics curricula and assessments offered by local districts and schools, which is needed for answering research question (II).

I decided to choose Texas as the target state. According to the National Center for Education Statistics (NCES, 2005), Texas has 8,746 elementary and secondary schools, 4,405,215 students, and 294,547 teachers, which makes it the second largest public education system in the US.

The current Texas state mathematics curriculum standard, Texas Essential Knowledge and Skills (TEKS), was initially issued by state Board of Education in September 1997. An amended version was approved in August 2006. To graduate from a public high school, a student needs to take three credits of mathematics courses, among which first-year algebra (i.e., Algebra I) is required (one credit). Other mathematics courses offered in high schools include second-year algebra (Algebra II, one-half to one credit), Geometry (one credit), Precalculus (one-half to one credit), and Mathematical Models with Applications (one-half to one credit). Although Algebra I is typically a ninth-grade course, more and more middle schools have started to offer it to their eighth grade students who have taken Pre-algebra.

The state education administration, Texas Education Agency (TEA), appoints a committee to review and recommend the textbooks from which districts and schools select. On the list of currently approved textbooks, the 16 for Algebra I were published between 1994 and 1998. At the time this dissertation was being written, the state was undergoing a new round of textbook review and adoption.

2. Mathematical backgrounds

There are two key, compounded factors in teachers' mathematical backgrounds: (a) college degree and major, and (b) mathematics and mathematics education courses taken during their professional preparation and development. The participant recruitment process did not allow enough time for me to gather detailed information about the potential participants' course-taking. Instead, information was collected on the following simpler attributes: (a) level of degree, major, and minor, (b) teaching certificate: subject areas and grade levels, and (c) school level (middle or high) at which the participant is currently teaching mathematics. By considering these three factors, I expected a high degree of variance among teacher characteristics.

3. Teaching experiences

The quantitative aspects of teaching experience include (a) the number of years one has taught mathematics in general and (b) the number of years spent teaching school algebra in particular. If an educator has never taught an algebra course, or has taught it before but not in recent years, he or she may not be familiar enough with the most current curriculum content and the major issues in teaching and learning. For that reason I decided to confine the population to instructors who have taught a first-year algebra course in the last three years or who are currently teaching such a course.

On the qualitative side, teaching experiences involve (a) the kind of courses the teacher has taught, (b) the textbooks the teacher has used, and (c) the professional development workshops the teacher has attended. Variance in each of these factors is likely to be guaranteed as long as enough teachers are recruited across school levels and school districts.

To guarantee the variance in teachers' backgrounds and experiences among the subgroups defined in research question (II) (mathematics major versus non-mathematics major, high school versus middle school groups), and also to render any statistical comparisons meaningful, I planned to recruit a minimum of 80 participants for the first phase of the data collection (background questionnaire and written-response assessment). Ideally, they would be distributed equally across the subgroups (Table 4.1):

Table 4.1 Proposed sample size by teacher subgroups

College major / School level	Mathematics major	Non-math major
High school	20	20
Middle school	20	20

Participant Recruitment

In early November 2006, I sent out recruitment emails to mathematics teachers in over 65 middle schools and high schools in 17 Texas school districts. Each email included (a) a brief description of the objectives of the study, what the participants were expected to do, the voluntary nature of the study, the risk, privacy and confidentiality issues, as well as the financial compensation for the participation ($70 for completing all questions and returning the instrument to the researcher); (b) a request for the participants' contact information at school; and (3) a very brief survey of their academic backgrounds and teaching experiences (number of years teaching mathematics and algebra, areas and grade levels of teaching certificates, and use of algebra textbooks). About 120 teachers responded, showed interest, and provided the requested information.

The second phase of participant recruitment occurred in January 2007, after the participants had completed and returned their written instruments in the first phase and a preliminary analysis had been conducted on their responses. Fifteen teachers were selected among them and formally invited to participate in a semi-structured follow-up interview. The strategy utilized was maximum variation sampling (Patton, 2002), which aims at capturing and describing the patterns and themes that cut across the variation in the participants' knowledge, conceptions, and experiences. These participants included both middle school and high school teachers, with different teacher preparation experiences and algebra teaching experiences. Their performance in the written-response assessment also varied. The plan was to select, eventually, eight participants of varying characteristics for the interviews (Figure 4.1):

- All participants (8)
  - Experienced (4)
    - Math major (2)
      - Higher performance (1)
      - Lower performance (1)
    - Non-math major (2)
      - Higher performance (1)
      - Lower performance (1)
  - Novice (4)
    - Math major (2)
      - Higher performance (1)
      - Lower performance (1)
    - Non-math major (2)
      - Higher performance (1)
      - Lower performance (1)

Figure 4.1 Participant recruitment for the follow-up interviews

THE PARTICIPANTS

Seventy-two mathematics teachers from Texas completed and returned the background questionnaire and written-response assessment. They work in 53 schools that are located in 11 school districts in central Texas and are all certified to teach mathematics at the K-8, 6-12, or 8-12 levels. Thirty-three of them are currently teaching mathematics in middle schools, and the other 39 are teaching in high schools.

1. College degrees

Exactly half of the participants had mathematics majors in college. Details of the participants' college degrees and their distribution are shown in Table 4.2:

Table 4.2 Summary of the participants' college majors

School level \ Major	Math	Math education	Education	Other	Total
Middle school	14	1	9	9	33
High school	22	4	2	11	39
Total	36	5	11	20	72

For these participants, "Other" majors include kinesiology, psychology, sociology, physics and other sciences, computer science, civil engineering, chemical engineering, interdisciplinary studies, French, music, and dance.

Forty participants have a minor degree from college (Table 4.3):

Table 4.3 Summary of the participants' college minors

School level \ Minor	Math	Sciences	Education	Other	Total
Middle school	8	3	1	5	17
High school	5	5	4	9	23
Total	13	8	5	14	40

The combinations of majors and minors vary greatly. The most popular combination for the middle school teacher group is an Education major plus a Mathematics minor (five participants). The most popular combination for the high school teachers is a Mathematics major plus a Science minor (four participants).

Twenty-seven participants have a master's degree or are in the process of earning one (Table 4.4):

Table 4.4 Summary of the participants' master's degrees

Master's program / School level	Math	Math education	Education	Other	Total
Middle school	1	2	6	1	10
High school	5	3	5	4	17
Total	6	5	11	5	27

Among those 10 middle school teachers, three have both college majors and master's degrees in Education. Of those 17 high school teachers, four have both college majors and master's degrees in Mathematics, and three have college degrees in Mathematics plus master's degrees in Education.

2. Teaching experiences

The participating teachers' total number of years of mathematics teaching ranges from 1 to 35 years, with an average of 10.4 years, a median of 9 years, and a mode of 7 years. Their total number of years of algebra teaching ranges from 1 to 30 years, with an average of 8.3 years, a median of 7 years, and a mode of 6 years. The histograms in Figure 4.2 and Figure 4.3 represent their years of mathematics teaching and years of algebra teaching, respectively.

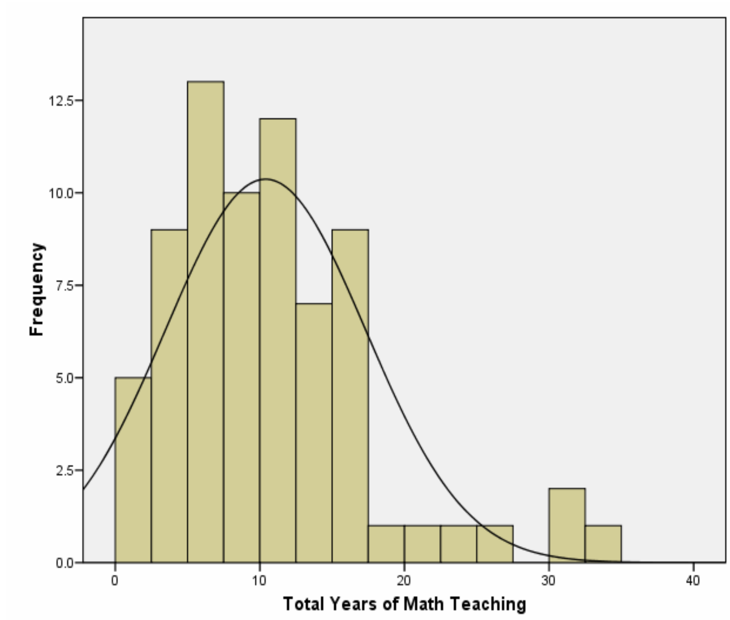


Figure 4.2 Histogram of the participants' years of mathematics teaching

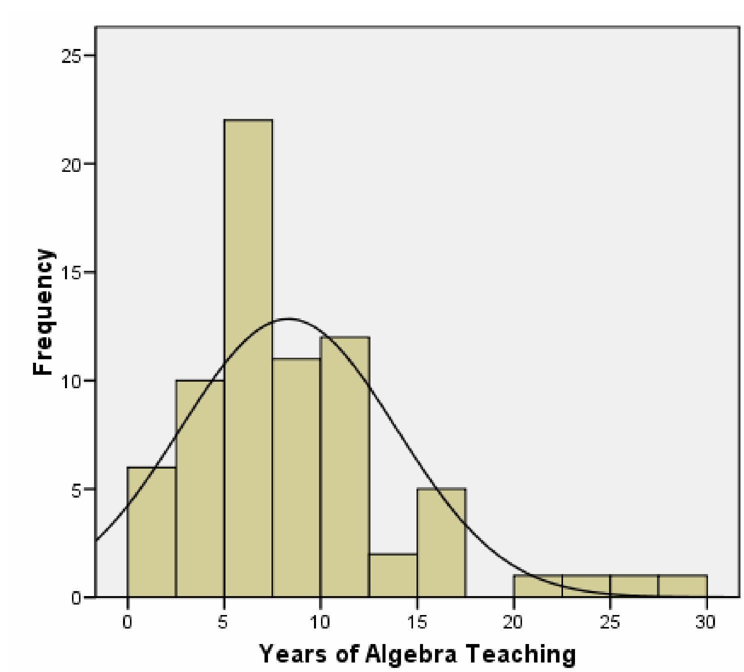


Figure 4.3 Histogram of the participants' years of algebra teaching

3. The interviewees

Eight teachers agreed to participate in the follow-up interviews. Their backgrounds and performance are close to the recruitment plan described in Figure 4.1 (p. 86). They are from six different schools in four districts. Seven of them are female, and one is male. Table 4.5 lists basic information about their education backgrounds, teaching experiences, and performance scores in the written-response assessment. All names are pseudonyms. The letters "H" and "M" refer to high school and middle school, respectively.

Table 4.5 Basic information about the interview participants

Teacher	School level	College major	College Minor	Master degree	Math course-taking	Years of math teaching	Years of algebra teaching	Score
Amy	H	Math	Spanish	Math	6	7	6	80
Jane	M	Math Ed	Economics	-	5	16	5	72
Mary	M	Math	-	-	6	7	7	42
Pam	M	Math	Astronomy	-	8	10	6	81
Rene	H	Sociology	Statistics	Math	7	5	3	53
Teresa	H	Kinesiology	Math	-	5	5	5	50
Tom	H	Math	Chemistry	Math	6	10	8	88
Yvonne	H	Math	Spanish	-	3	15	15	50

RESEARCH INSTRUMENTS

Three basic types of research instruments and techniques were designed and employed in this study: an academic background questionnaire, a set of written-response assessment questions, and a semi-structured interview protocol. In the design process, I paid attention to the issues of content validity and construct validity by following two basic principles: (a) centering the instruments on the research questions and (b) selecting and structuring the items in accordance with the conceptual framework.

Table 4.6 indicates the alignment between the research questions and the techniques:

Table 4.6 Alignment between research methods and research questions

Methods \ Research Question	I			II	
	I.1	I.2	I.3	II.1	II.2
Academic background questionnaire				√	√
Written-response assessment	√	√		√	√
Semi-structured interview protocol	√	√	√		√

Academic Background Questionnaire

This two-page questionnaire (Appendix 1) includes eight questions designed to elicit quantitative and quantifiable information about teachers' professional backgrounds and experiences, which was needed for answering research question (II) and its two specified subparts. The questions appear in three formats: fill-in-the-blanks, partial open-ended questions (multiple-choice with an "Other" option that one could fill in), and multiple-choice questions (Table 4.7).

The questionnaire preceded the set of written-response assessment questions, and the expectation was that it would be completed in about five minutes.

Table 4.7 Formats of the background survey questions

Question	Focus	Format
1	Total number of years of mathematics teaching	Fill-in-the-blank
2	Degrees earned and major, minor	Fill-in-the-blank
3	Areas of certification and grade levels	Fill-in-the-blank
4	Course-taking in advanced mathematics and mathematics education	Partial open-ended
5	School algebra courses taught	Partial open-ended
6	Number of years of algebra teaching	Fill-in-the-blank
7	Algebra textbooks used	Partial open-ended
8	Participantion in professional development activities	Multiple-choice

Written-response Assessment

The written-response assessment questions (Appendix 2) were designed to generate data about teachers' mathematical knowledge for teaching algebraic equation solving, which was essential for answering both research questions (I) and (II). These questions follow two main formats: multiple-choice questions and open-ended questions.

Most of the multiple-choice questions in this assessment were written to measure the first domain of teachers' mathematical knowledge for teaching, knowledge of the subject matter. Partially, this is because it is possible to design questions to which each answer could be unanimously determined as right or wrong purely by clear mathematical criteria. Below is an example (assessment question 5.3):

5.3 If $ax + b = 0$ and $cx + d = 0$ are two **equivalent** but **different** linear equations, what can we say about the two corresponding lines $y = ax + b$ and $y = cx + d$? Please determine the truth of each of the following statements:

	Always true	Possibly true	Impossible
1) These two lines are identical	☐	☐	☐
2) These two lines are parallel	☐	☐	☐
3) These two lines have the same x-intercept	☐	☐	☐
4) These two lines have the same y-intercept	☐	☐	☐
5) These two lines are perpendicular	☐	☐	☐

Most of the assessment questions were written in open-ended forms. The major advantage of using open-ended items is that they allow the researcher to gather detailed information on teachers' thinking and the reasoning behind their answers to a specific question, which may not have been documented by previous studies.

Content-wise, the assessment questions focus mainly on linear and quadratic equations. As disclosed by the textbook survey in Chapter 2, solving linear and quadratic equations is a common topic for various first-year algebra textbooks. Hence, it is reasonable to assume that most middle school and high school algebra teachers would be familiar with these topics.

In the assessment instrument there are six main items, and each includes three to six specific questions. Besides addressing the three domains of mathematical knowledge for teaching, the questions were designed to cover all three aspects of mathematical procedures characterized in the framework: (a) basic algorithms, (b) alternative algorithms and strategies, and (c) related concepts, procedures, and properties. For each aspect, two or three fundamental topics were identified through literature and textbook reviews, and specific assessment questions were, correspondingly, designed:

1. Basic algorithms for solving equations

- The balancing method for solving linear equations (Item #1)
- The quadratic formula and completing the square method (Item #4)

2. Alternative algorithms and strategies

- The “undoing” method for linear equations (Item #2)
- The factoring method for quadratic equations (Item #3)
- Solving equations by graphing (Item #6)

3. Related concepts, procedures, and properties

- Equivalent equations, transformations, extraneous, and lost roots (Item #5)
- Solutions to linear equations and systems of linear equations (Item #6)

Semi-structured Interviews

The use of interviews as a data collection method bears the assumption that the participants’ knowledge and perspectives are meaningful, knowable, able to be made explicit, and relevant to the questions being studied (Miles & Huberman, 1994). Researchers such as Desimone and Le Floch (2004) have demonstrated that cognitive interviews can be a useful method for improving the validity and reliability of surveys used in educational research. For the present study, the semi-structured interviews could help to answer better research question (I) and part of research question (II) through eliciting in-depth information about the participating teachers’ knowledge, thinking, reasoning, and experiences that was not fully revealed or covered by the questionnaire and written assessment questions.

The interview protocol (Appendix 3) includes eight sets of questions that focus on five algorithms and strategies for solving linear equations: (a) the balancing method, (b) the undoing method, (c) tracing points on the line graph, (d) examining the numerical table, and (e) finding the intersection of two lines. The specific focus for each set is as follows:

Question set 1: General issues regarding teaching basic and alternative methods for solving linear equations

Question set 2: The balancing method

Question set 3: The undoing method

Question set 4: Solving by intersecting two lines

Question sets 5, 6, and 7: Overall comparisons of the five methods

Question set 8: Teaching mathematical procedures and rules

The rationale for the sequencing of the question sets is as follows: The first set of questions draws the participant into the issue and elicits general information about their overall experiences and preferences. Question sets two to four probe for details in their knowledge and conceptions of three strategies for solving linear equations. After the participants have become familiarized with and thought about each individual strategy, the next three sets of questions examine how the participants relate and contrast the five methods. After answering all previous questions, the participants would be more prepared to answer the last set of questions, which is the most overarching and conceptual.

The interview questions also align with the conceptual framework: In one respect, they aim to provide clues for answering the research questions regarding teachers' understanding of the features and roles of each method, the related concepts and procedures, and the relationships among them. And in another respect, each set of questions addresses the three domains: (a) knowledge of the subject matter, (b) knowledge of learner conceptions, and (c) knowledge of didactic representations.

After the interviewees answered questions about each of the five methods, they were asked to rate each of these methods (on a one-to-five scale) by eight basic features: (a) accuracy, (b) generality, (c) efficiency, (d) mathematical value, (e) transparency, (f) easy to apply, (g) easy to teach, and (h) easy to learn. The resulting matrix provides quantitative data that reflect teachers' knowledge for and conceptions of teaching linear equation solving. Chapter 5 will summarize the data.

INSTRUMENT REVIEW, PILOT, AND REVISION

To increase the content validity of the instruments, the drafts of the background questionnaire and the written assessment questions were reviewed and revised through three phases: peer review, expert review, and small-scale pilot studies. The interview protocol draft was also piloted with a few teachers who completed the pilot questionnaire and assessment, and revised based on the pilot results as well as preliminary findings from the formal written-response assessment.

Peer Review

In April 2006, I invited six mathematics education graduate students and one mathematics graduate student, all at Michigan State University, to answer and then evaluate the initial draft of the written-response assessment questions. Five of the mathematics education graduate students have had mathematics teaching experiences in secondary schools in the US. Along with the assessment questions, I provided a review sheet for the reviewers to record their comments and suggestions on the following aspects:

- Timing: how long it took to answer all questions
- Level of relevance and importance of each question to the teaching and learning of algebraic equation solving
- Whether there are better ways to phrase each question
- Whether there are inaccuracies, ambiguities, or mistakes in any items
- Whether there are more important questions to ask for each topic

Based on the answers, comments, and suggestions provided by the reviewers, I modified a question on the balancing method, combined two items about equivalent equations, and removed the questions about the cover-up method. These changes allowed the instrument to be more focused on the key methods for solving equations and more aligned with the conceptual framework. I also revised the wording of most of the questions.

First Pilot Study

In May 2006, using the initial draft of the academic background questionnaire and the revised version of the written-response assessment questions, I conducted a pilot study with 12 high school mathematics teachers. The participants came from five high schools in Michigan with an average of 13 years of mathematics teaching experience and an average of nine years of algebra teaching experience.

Through examining teachers' responses and feedback, I was able to see whether each question could be correctly interpreted by the participants and could elicit thoughtful answers. I also got a better idea of the average time needed to complete each item and the entire instruments. Further, I started to generate an initial set of possible answers for the open-ended questions, which was essential for developing the coding and scoring rubrics.

Expert Review

In July 2006, I invited two experts in mathematics teacher education to review the second draft of the academic background questionnaire and written-response assessment questions. One of the reviewers is a mathematician who has been supervising undergraduate students enrolled in elementary and secondary mathematics teacher preparation programs in Texas, and designing and instructing mathematics content courses for those teachers. The other reviewer is a mathematics educator devoted to developing in-depth materials for mathematics teacher preparation and professional development programs across the country. Both of them provided comments on content coverage, relevance, and clarity of the questionnaire and assessment items.

In the revision that followed the first pilot and expert review, several questions were removed because they were very time-consuming, were not closely related to the research focus, or did not elicit details in teachers' thinking. A few new questions were added regarding students' conceptions and teachers' strategies for improving student understanding. Meanwhile, I added two new items about strategies for solving quadratic equations: the factoring method and the quadratic formula.

Second Pilot Study

In September 2006 I administered the revised written instruments to five mathematics teachers in Michigan as a second pilot. Afterward, three of these teachers agreed to participate in the pilot interviews. Each interview lasted about two hours.

Besides providing clues for the final revision of the written-response instruments, the second pilot helped me to restructure the interview questions in two ways: (a) reducing the number of equation solving topics so that the interview could be more focused, and (b) asking more specific questions, rather than general ones, so that teachers' answers could be more relevant and in-depth.

MAIN DATA COLLECTION

Data collection in this study underwent two phases, each following one corresponding phase of the participant recruitment.

The first phase consisted of data collection through the written instrument booklet, which included the background questionnaire and the written-response assessment questions. Among the 120 mathematics teachers in Texas who responded to the recruitment email and showed interest in participation, 100 were selected and formally invited to participate in the study. In November 2006, written instrument booklets were sent to the selected participants, together with a consent form, and a research subject payment request form. By mid-December 2006, a total of 72 teachers had completed and returned their instruments.

The second phase was the follow-up interviews. In early February 2007, 8 of the 15 teachers who were invited for the interviews agreed to participate. I traveled to Texas in late February to conduct the interviews. Each participant was interviewed for up to two hours at the participant's own school office, classroom, or library. A signed consent form was obtained before each interview began. As they all agreed to it, the interviews were audio recorded. Besides the audio files, two other types of data were also created or collected during each interview: (a) the participant's written responses, sketches, and rating matrix and (b) the notes I took during the interview.

DATA ANALYSIS

As mentioned in the previous sections, preliminary analysis on data collected from the first phase of the study preceded and informed the instrument design and data collection in the second phase. Once the interviews were completed, data gathered from all sources were analyzed systematically and thoroughly, and constantly checked against the research questions and the conceptual framework.

Overall, there were three different levels of analysis:

1. Item and instrument analysis: coding and scoring the participants' responses to the questions in a particular instrument, and summarizing codes, scores and themes for each instrument. Analyses at this level laid foundation for the category-level analysis.

2. Category-level analysis: linking and summarizing codes, scores, and themes across the instruments for each of the three domains of mathematical knowledge for teaching. These analyses involved both quantitative and qualitative methods and were the key analyses for answering the specific research questions.

3. Global analysis: connecting and comparing codes, scores, and themes from all instruments in a holistic manner, both quantitatively and qualitatively. These analyses were meant to yield overarching findings related to the research questions.

Item and Instrument Analysis

1. Academic background questionnaire

The numerical information obtained through the questionnaires (e.g., number of years of teaching) was recorded directly. All other information was first converted into numerical codes, then recorded. For example, in coding the participants' college majors or minors, 1, 2, and 3 were used to refer to Mathematics, Mathematics Education, and Education, respectively.

2. Written-response assessment

For each multiple-choice question, one cell was created in the database file for each of the answer options. Whenever an option was selected, the code "1" was entered into the corresponding cell. Otherwise, the cell was left blank. When appropriate, another cell was created for the participant's score on this question.

For the open-ended questions, a crucial step was to develop rubrics for coding and scoring the responses. Each rubric typically included two parts:

- A qualitative categorization of the nature or focus of a particular response. For some questions, such categorization centered on the exact three domains of knowledge defined in the conceptual framework: knowledge of the subject matter, knowledge of learner conceptions, and knowledge of didactic representations. The codes “M”, “L”, and “R” were used to represent these three categories, respectively. For some other questions, the categories were summarized from the participants’ varied responses.

- A quantitative score that measured the relative level of appropriateness or reasonableness of a response. Based on a holistic-scoring method introduced by Thompson and Senk (1993, 1998), a generic scoring scheme was developed for this written-response assessment:

3 points – The answer is complete, clear, most reasonable, and most relevant to the question.
2 points – The answer is complete, but there are some minor problems with its clarity, reasonableness, or relevance.
1 point – The answer is incomplete, or there are some major problems with its clarity, reasonableness, or relevance, but there is at least one correct or reasonable thought.
0 points – There is no answer, or the answer is completely incorrect, meaningless, or irrelevant.

When being applied to each particular question, the generic rubric was customized into a more specific version that fit better with the nature and format of that question. Take assessment question 1.2 as an example:

1.2 In solving linear equations with the balancing method, what major types of mistakes (other than computational errors) or difficulties have you seen from students? Please list two different types of mistakes or difficulties, and correspondingly provide strategies for helping students to improve their understanding.

The mistakes or difficulties that participants described when answering the first part of the question were categorized, as follows, by the nature of the problem in student learning:

1 – Problems with negative sign, subtraction, or additive inverse
2 – Imbalanced operations on the two sides
3 – Combining unlike terms
4 – Problems with multiplicative inverse
5 – Misuse of the distributive property
6 – Problems with order of operations
7 – Problems with fraction coefficients
8 – Others

Similarly, responses to the second part of the question (i.e., strategies for improving student learning) were categorized as follows:

1 – Using visual representation, hands-on tools, or metaphors to explain
2 – Emphasizing the meaning of a concept, symbol, or property
3 – Asking the student to double-check answers, verifying answers with technology, or considering counterexamples
4 – Analyzing the process (showing chart of operations and inverses; doing one step at a time, discussing when each step is appropriate and why, or why certain processes lead to wrong results)
5 – Modeling or directly showing the rules or processes, providing alternative examples
6 – Reviewing or re-teaching concepts, methods, and rules previously taught
7 – Practice and drill

An individual participant’s responses to these two parts were paired and scored as a whole, based on the following rubric:

3 points – Legitimate mistake or difficulty, with a most clear, relevant, and reasonable solution
2 points – There are minor problems in the legitimacy of the mistake given, or in the clarity, relevance, and reasonableness of the strategy
1 point – There is a major problem in the legitimacy of the mistake given, or in the clarity, relevance, and reasonableness of the strategy
0 points – Either the mistake, the strategy, or both are not legitimate.

After all participants' responses to a certain question were coded and scored, the frequency of the categorical codes was then computed, and the quantitative scores would later be combined with those from other questions.

To assure the reliability of the coding and scoring, I asked for help from three other raters who are doctoral students in the mathematics education and teacher education programs at Michigan State University. I first shared with them the draft rubrics I developed, as well as 20 sample responses. After discussion, we agreed to make some revisions to the rubrics. At that point, I scored all 72 sets of responses, and the other three raters independently scored items 1 and 2, items 3 and 4, and items 5 and 6, respectively. My own scores were then compared with those of each of the three raters, and inter-rater reliabilities were computed. The inter-rater agreement on items 1 and 2 was 72%, on items 3 and 4 was 75%, and on items 5 and 6 was 78%. When there was disagreement on the coding or scoring of a certain item, I compared the two scores and decided the final score.

For assessment questions 1.1 and 2.4, the participants' responses were too diverse to be evaluated easily by holistic rubrics. I recorded all the typical answers, categorized them based on the framework, and sent them to three mathematics education experts for ratings. They rated all the typical answers and provided comments on the categorizations and ratings. Finally, the three sets of ratings were averaged to become the final score for each given response.

3. Semi-structured interviews

Right after each interview, I typically read quickly through my notes and the artifacts collected from the interviewee to make remarks, to make note of potential themes or questions, and to summarize my reflections. Later, I transcribed the audio records from all eight interviews and entered the interviewees' rating matrices into the database.

Developing some manageable classification and coding scheme is the first step of analyzing interview data (Patton, 2002). The transcripts for each interview were analyzed across interviewees through focused coding, with the goal of generating overarching themes. For example, two basic themes emerged among the interviewees' preferences for teaching different strategies for solving linear equations: teaching the balancing method only and teaching more than one method. When asked about the reasons for teaching multiple methods, those participants who do teach more than one method provided two major justifications: (a) Learner-oriented: Students learn differently. For each student, some methods are easier to understand than others. (b) Method-oriented: Different methods work well on different types of equations. Students should be able to choose flexibly the best strategy when solving a given equation.

Interview question set 4 was task-based. The focus of the analysis was to find out how the interviewees approached the problem (e.g., graphically, symbolically, or case-based) and whether or not they could see the connection between the graphical and symbolic aspects of the balancing process. The depths of their reasoning as well as the degrees of flexibility in their switching between different approaches were compared.

Question set 6 called for a rating matrix from each interviewee on the five methods for solving linear equations. The matrices were entered as numerical tables in Microsoft Excel. Basic computations were then performed by columns (i.e., the five methods) and rows (i.e., the eight attributes of a mathematical procedure), and within and across tables (i.e., the interviewees) to produce (a) the weights that each interviewee assigns the five methods and eight attributes, (b) the eight interviewees' average ratings of each of the methods and attributes, and (c) the methods and attributes that were rated as the most similar or different. When cross-referenced with the interviewees' think-aloud explanations during the rating processes, these computations could be quantitative measures of the interviewees' individual and overall understanding of the different methods and their major attributes.

Category level and global analysis

Analyses at the category and global levels are the direct means for answering the two main research questions and their specified versions. A few steps were undertaken in the process of these analyses:

1. Each participant's final scores on individual items and the three domains of knowledge (subject matter, learner conceptions, and didactic representations) were summed. These total and subtotal scores represented quantitative measures for each participant's overall knowledge and specific knowledge of individual domains and aspects.

2. A *descriptive statistical analysis* was applied to the above scores for all of the participants, which produced minimal, maximal, and mean scores; standard deviation and standard error mean values; as well as histograms.

3. *Hypothesis testing* was performed to contrast the average scores of different participant groups, mainly, mathematics versus non-mathematics majors, number of mathematics courses taken, and longer versus shorter algebra teaching experiences. The Bonferroni method was selected for preplanned comparisons and for controlling errors associated with statistical inferences.

4. *Exploratory analyses* were performed to find out if there was significant correlation (a) between the participants' college major and mathematics course-taking, and (b) between course-takings in mathematics and in mathematics education.

5. Based on the codes and themes generated at the item level, the interview data were further examined holistically across all questions for each interviewee and also across interviewees.

6. Results from the first four steps were related to and contrasted with the data, results, and themes from the interview analysis in conclusion drawing and verification. Major consistencies and inconsistencies were scrutinized and discussed.

Chapter 5 Results and Discussion

OVERVIEW

As indicated in the previous chapter, three types of data have been produced through the data collection processes: (a) the participating teachers' responses to the academic background questionnaire, (b) their responses to the written assessment questions, and (c) the notes and recordings of the semi-structured interviews with eight selected teachers. They were analyzed with a blend of qualitative and quantitative methods. This chapter summarizes and discusses the major results of this data analysis. Aligned with the two main research questions and their sub-questions, this chapter is organized into two main sections with a few sub-sections in each:

1. The status of the participating teachers' mathematical knowledge for teaching equation solving.

This section addresses their overall performance, their responses to different kinds of questions regarding the teaching and learning of equation solving, their conceptions of the features and relationships among multiple strategies for solving linear equations, and their conceptions of the role of mathematical routines.

2. The relationships between the participants' mathematical knowledge for teaching equation solving and a few basic variables in their academic backgrounds and teaching experiences.

Some of these variables include college major, advanced mathematics and mathematics education course-taking, school algebra course-teaching, and number of years of teaching algebra. Teachers' own accounts of the sources of impact on their knowledge and conceptions are also summarized.

TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING EQUATION SOLVING

The status of teachers' mathematical knowledge for teaching equation solving is analyzed quantitatively by teachers' overall performance and scores on each item, and based qualitatively on their responses to the assessment and interview questions.

Descriptive Statistics of Overall Performance

The participants' total scores on the written-response questions range from 19 to 88 points (out of a full score of 109 points), with mean score $\mu = 54.6$, median $\mu_{1/2} = 54.5$, standard deviation $\sigma = 16.5$, and standard error mean $S_E = 1.9$. Figure 5.1 is the histogram of the total scores:

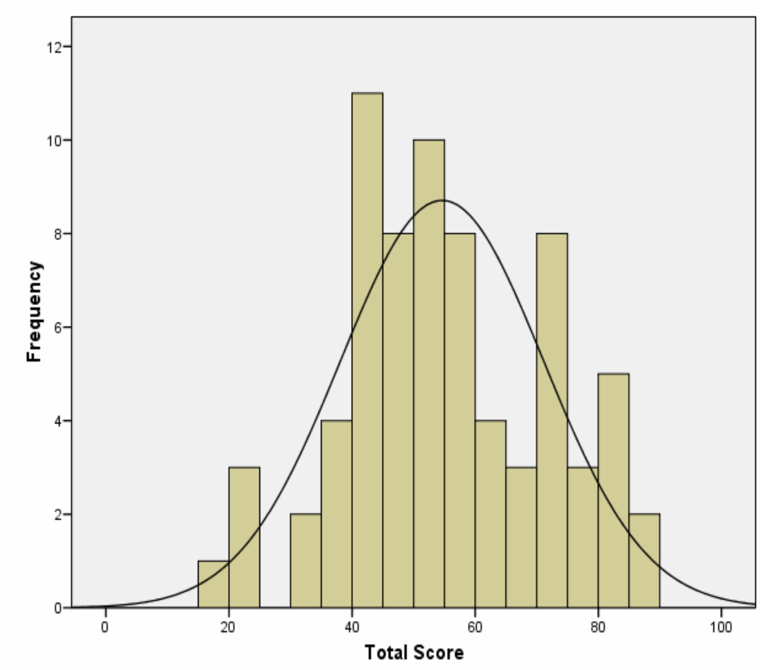


Figure 5.1 Histogram of total scores

Teachers' Responses to Subject Matter Knowledge Items

Descriptive statistics

In the written-assessment instrument, seven questions focus directly on some in-depth subject matter knowledge that teachers may need for effectively teaching equation solving. The participants' subtotal scores on those questions range from 2 to 42 points (out of a full score of 47 points), with mean score $\mu = 21.8$, median $\mu_{1/2} = 21.0$, standard deviation $\sigma = 8.1$, and standard error mean $S_E = 0.9$. Figure 5.2 is the histogram of the subtotal scores:

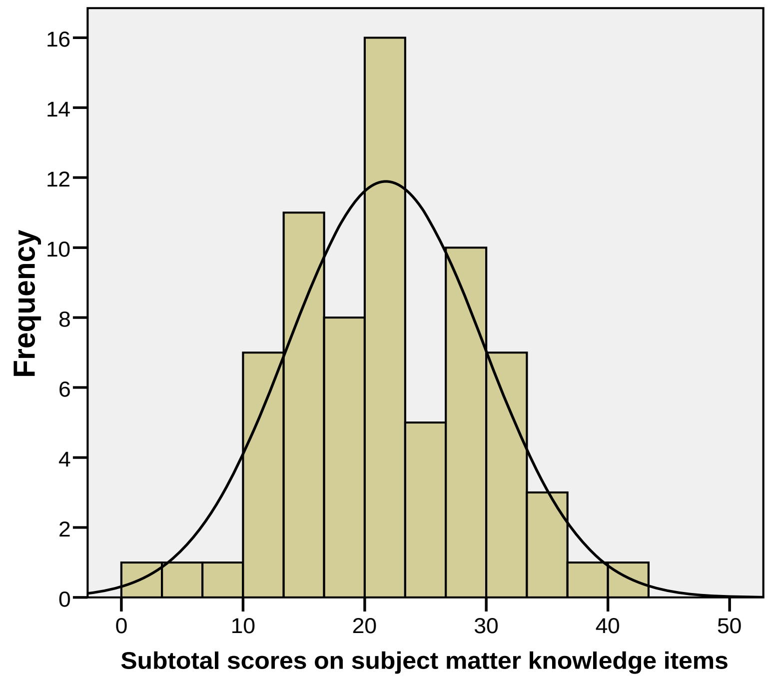


Figure 5.2 Histogram of subtotal scores on subject matter knowledge

Applicability of a certain method

In teaching a certain mathematical method, teachers may need to go beyond knowing how to apply the method to typical problem situations and understand whether the method is generalizable to other types of problem (i.e., be clear about the range of problems to which the method is applicable). In the written-assessment, a few questions are asked about the applicability of the undoing method and the intersecting-the-line method for solving linear equations.

1. The undoing method

Question 2.2 and 2.3 address the applicability of the undoing method:

2.2 On which kinds of **linear equations** can we directly apply the undoing method?

2.3 Does this method work for non-linear equations? What would be the characteristics of the kind of equations that can be solved directly by this method?

For question 2.2, 15 (20.8%) of the participants answered correctly and accurately by stating that the linear equation should be "in the form of $ax + b = c$," or one "with a single variable term," or "one-step and two-step equations." Thirty (41.7%) of the participants' answers had minor or major problems in accuracy or clarity (for example, "those in slope-intercept form," "simplified," "solve for one variable," and "those without the variable on both sides").

The remaining 27 (37.5%) participants gave incorrect answers. Most of them claimed that the method can be directly applied to "any" or "most" kinds of linear equations. The follow-up interviews revealed a potential explanation for such kind of answers. Six of the eight interviewees thought that the undoing method was essentially the same as or very similar to the balancing method, since both processes involve undoing or inverse operations. They did not realize that the undoing method views one side of the equation as a sequence of operations on the unknown and the other side as the final result. The solving process has to begin with the constant and undo each operation in the sequence in reversed order. In other words, the unknown variable can only appear once in the equation, and its value will be figured out only in the very last step.

For question 2.3, 40 (55.6%) of the participants either believed the method worked for linear equations only or went the opposite direction by stating that it works for all kinds of equations. Twenty-eight (38.9%) were not precise enough with their answers or gave only specific examples of non-linear equations that could be solved by undoing. Only five (6.9%) participants had completely correct answers (i.e., the method works on those non-linear equations in which the variable appears only once and which involves only one-to-one functions).

Among the 27 participants who answered question 2.2 incorrectly, 19 also gave wrong answers to question 2.3, and another six gave answers with major problems.

Before answering questions 2.2 and 2.3, the participants were asked to indicate whether they have ever taught the undoing method. Twenty-four (33.3%) answered "yes," and 48 (66.7%) answered "no." The two groups' performance on question 2.2 are compared in Table 5.1 and Figure 5.3.

Table 5.1 Summary of responses to Question 2.2 by teaching experience

Participants / Score on Question 2.2	Have taught the method		Have not taught the method	
	Frequency	Percentage	Frequency	Percentage
3 points	6	25	9	18.8
2 points	6	25	8	16.7
1 point	4	16.7	12	25
0 points	8	33.3	19	39.6
Total	24	100	48	100

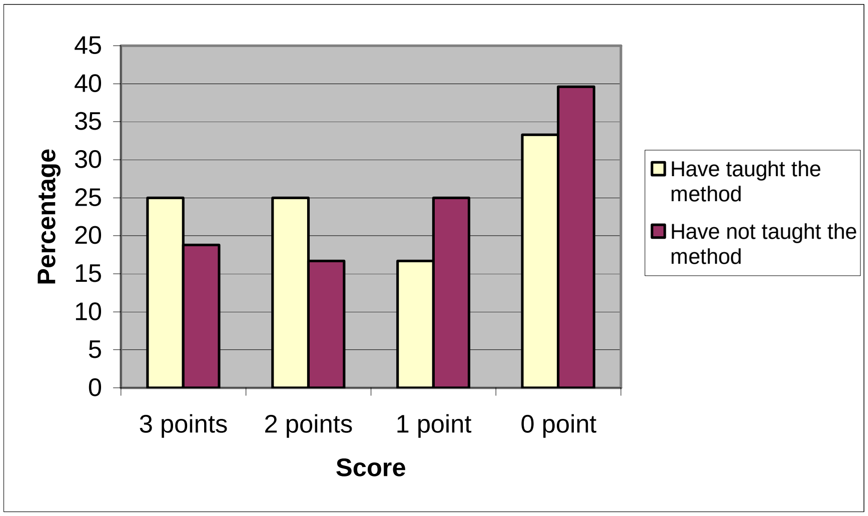


Figure 5.3 Distribution of scores on question 2.2 by participants' experience

The two groups' scores on question 2.3 are shown in Table 5.2 and Figure 5.4:

Table 5.2 Summary of responses to Question 2.3 by teaching experience

Participants / Score on Question 2.3	Have taught the method		Have not taught the method	
	Frequency	Percentage	Frequency	Percentage
3 points	3	12.5	2	4.2
2 points	3	12.5	4	8.3
1 point	8	33.3	14	29.2
0 points	10	41.7	28	58.3
Total	24	100	48	100

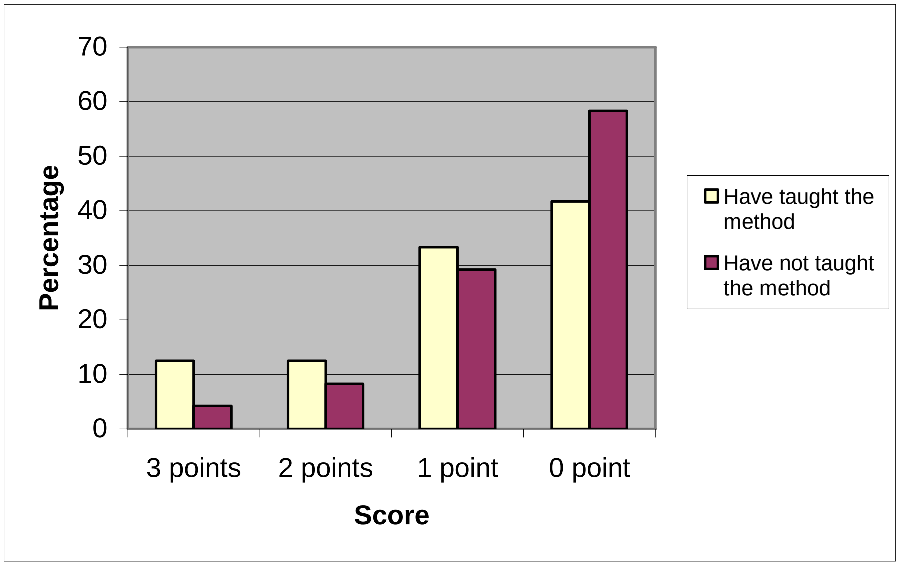


Figure 5.4 Distribution of scores on question 2.3 by participants' experience

The above results show that, overall, teachers who had taught the undoing method understood the issues better than those who had not taught it. But, for both groups, the highest percentages were those who got the answers to the questions completely wrong.

2. The intersecting-the-lines method

After the intersecting-the-line method (finding the *x*-coordinate of the intersection of $y = f(x)$ and $y = g(x)$ to solve a linear equation $f(x) = g(x)$) was introduced, question 6.3 asked about the applicability of the method:

6.3 Besides linear equations, for what other types of equations would this method also work?

The participants' responses by score are shown in Table 5.3:

Table 5.3 Summary of responses to Question 6.3

Score	Frequency	Percent
3	13	18.1
2	4	5.6
1	33	45.8
0	22	30.6
Total	72	100.0

Those participants who got 3 points realized that this method applies to any equation $f(x) = g(x)$ where $f(x)$ and $g(x)$ are functions. Those responses that scored two points included such descriptons as, "It applies to most types of equations," "all except rational cause the restricted domain," and "all continuous functions." Those that scored one point only covered a specific type of equation, such as quadratics, exponential, or "linear on one side, quadratics on the other." The rest of the 22 (30.6%) participants got zero points because they claimed the method only applied to linear equations, wrote "I don't know," or did not provide any responses.

Conditions and constraints on a method or property

In mathematics, a method or property that works in a certain system may no longer be valid when the system is changed or some of the constraints on the method or property are altered. The four properties of equality state that equality still holds when the same *number* is added to, subtracted from, multiplied on, or divided from both sides of the equality (non-zero for division). These properties are the theoretical foundation for the balancing method for solving linear equations. However, one has to be careful when adding, subtracting, multiplying, or dividing an *expression* on both sides of an equation, which happens more often for solving non-linear equations. Question 5.2 in the assessment targets this issue:

5.2 Does each of the following transformations on an equation **always** generate an equivalent equation? Give an example if you choose "no."

	Yes	No	Given an example if you choose "no"
1) Adding $\sqrt{x}$ on both sides	☐	☐	
2) Multiplying the two sides by (x +5)	☐	☐	
3) Squaring both sides	☐	☐	
4) Taking square roots of both sides	☐	☐	

All of the above four cases may generate non-equivalent equations, since each of the transformations could cause change in the domain of the functions involved. Table 5.4 shows, for each case, the frequency and percentage of answers that are fully correct (the participant selected "no" and also gave a right example) or completely wrong (the participant either selected "yes," or selected "no" but gave a wrong example or no example).

Table 5.4 Summary of responses to Question 5.2

Case / Answer	1)		2)		3)		4)	
	Count	Percent	Count	Percent	Count	Percent	Count	Percent
Fully correct	4	5.6	12	16.7	13	18.1	4	5.6
Completely wrong	67	93.1	56	77.8	51	70.8	60	83.3

Only one participant (1.4%) answered all four questions correctly.

For case 1), several participants selected “no” but explained that “$\sqrt{x}$ can be positive or negative” without realizing the real problem with $\sqrt{x}$: its domain is non-negative numbers, which may cause the loss of roots to the original equation. Similarly, a few participants answered “no” for case 4) with the explanation that “square roots can be positive or negative,” while the real problem is the possibility of having lost roots because of the constraints put on the domain of the variable by the square root function.

Figures 5.5 and 5.6 show two algebra teachers’ misconceptions in responding to Question 5.2. The teachers oversimplistically applied the four properties of equality without realizing the constraints.

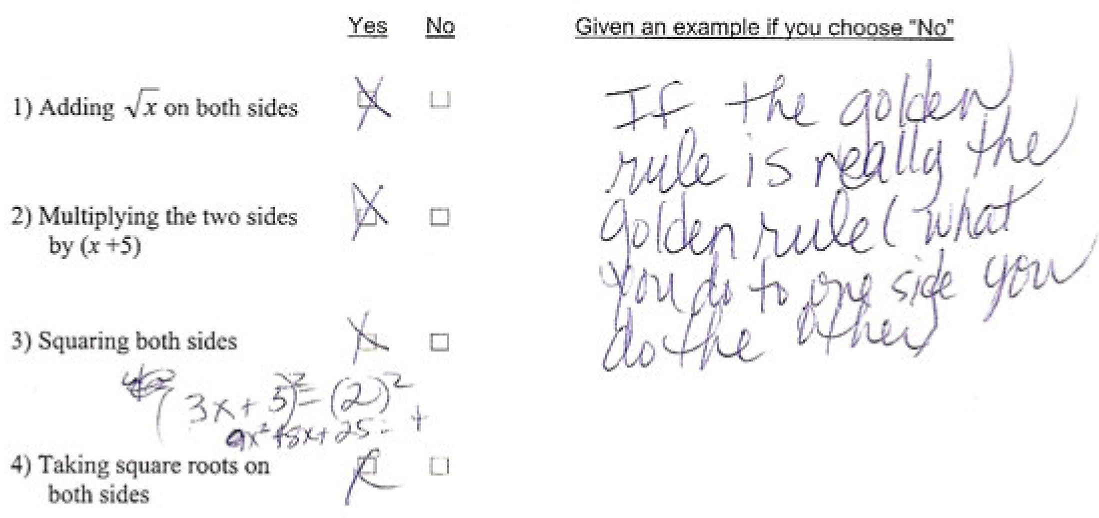
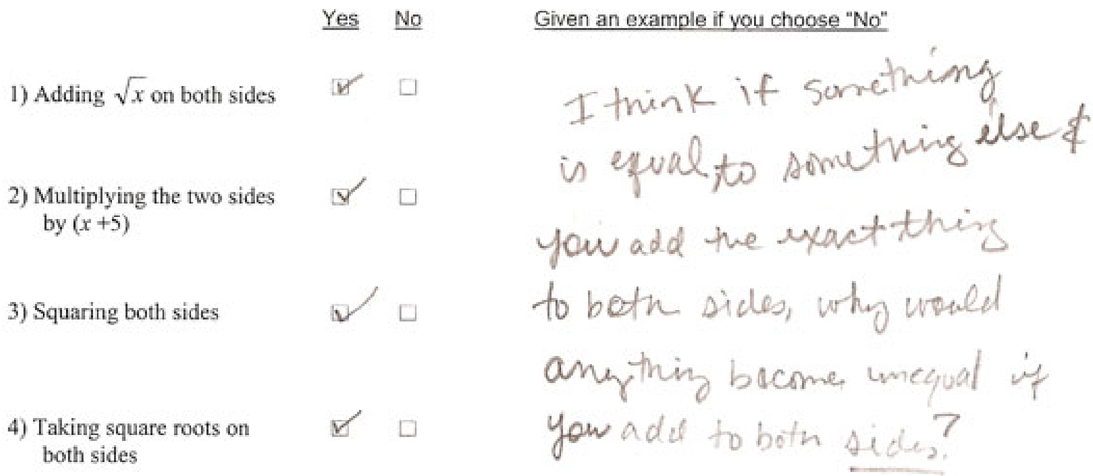


Figure 5.5 A sample response to Question 5.2



	Yes	No	Given an example if you choose "No"
1) Adding $\sqrt{x}$ on both sides	☑	☐	I think if something is equal to something else & you add the exact thing to both sides, why would anything become unequal if you add to both sides?
2) Multiplying the two sides by $(x+5)$	☑	☐	
3) Squaring both sides	☑	☐	
4) Taking square roots on both sides	☑	☐	

Figure 5.6 A sample response to Question 5.2

The participants' low performance on this item could be attributed to two causes: (a) the issues of extraneous roots and lost roots are mostly dealt with in second-year algebra, so many first-year algebra teachers (especially those from middle schools) may not be familiar enough with them, and (b) textbooks teach how to avoid extraneous or lost roots through answer-checking after the solving processes. As a habit of mind, teachers and students would have been more sensitive to the effects of various transformations if the domain of the variable could have been addressed before and during the solving process

Prerequisite knowledge and skills for a method

To unfold the conceptual underpinnings of a mathematical method, teachers need to be very clear about what prior mathematical concepts, processes, and properties are essential to the establishment of the method. The written assessment includes two such questions on solving quadratic equations, one on the factoring method and the other on the quadratic formula.

1. The factoring method

Question 3.1 probes teachers' knowledge of the prerequisites to the factoring method:

3.1 Which of the following algebra knowledge and skills are essential for students to understand fully the factoring method? Check all that apply.

☐ **A.** Combining like terms in an expression
☐ **B.** Multiplying two binomials
☐ **C.** The distributive property
☐ **D.** The zero-product property
☐ **E.** Solving linear equations of the form $ax + b = 0$

Factoring a trinomial and multiplying two binomials are two sides of the same coin: the distributive property. To make sense of the factoring process, one has to understand the multiplication and combination process. Therefore, all of the five choices indicate prerequisite knowledge and skills for the factoring method. The number and percentage of participants who correctly selected each choice are as follows:

A. 66 (91.7%)

B. 69 (95.8%)

C. 67 (93.1%)

D. 51 (70.8%)

E. 43 (59.7%)

Overall, a total of 32 participants (44.4%) correctly selected all five choices.

Nearly 30% of participants did not choose D. One possible explanation could be that some teachers do not use or are not familiar with the name of the property, regardless of whether they emphasize the property or not.

In neglecting choice E, some participants may simply have thought that solving linear equations and solving quadratic equations were not directly related, or forgotten that two linear equations emerge and each needs to be solved when a quadratic equation is factored into $(ax + b)(cx + d) = 0$.

2. The quadratic formula

The quadratic formula is derived from another general method: completing the square. Question 4.1 assess teachers' understanding of this matter:

4.1 Through what major methods or strategies is the quadratic formula derived?

Thirty-five participants (48.6%) correctly pointed out completing the square as the major method underlying the derivation of the quadratic formula, while 33 (45.8%) provided completely false answers or no answer.

Mathematics connections among methods and concepts

Connectedness among one's knowledge of different concepts and methods is a core measure of his or her conceptual understanding. Specific to solving linear equations, the balancing method, the concept of equivalent equations, and the function-based intersecting-the-line method are closely tied together.

1. The balancing method and the concept of equivalent equations

When we use the balancing method to solve a linear equation (i.e., performing transformations or using four basic operations) on the two sides, each step typically produces an equivalent equation. Therefore, the other way around, we could possibly determine the equivalence of two linear equations by examining whether they are linked by a transformation. Question 5.4 is intended to assess related teacher knowledge:

5.4 For each of the following pairs of linear equations, determine whether the two equations are equivalent or not, **without actually solving the equations.** Please explain your reasoning.

	Pair of equations	Are they equivalent?	Your reasoning (without solving the equations)?
1)	$3x - 4 = 16$ and $3x - 7 = 13$	☐ Yes ☐ No	
2)	$2x + 8 = 4x - 15$ and $3x + 9 = 4x - 14$	☐ Yes ☐ No	
3)	$2x - 4 = 3x + 16$ and $4x - 7 = 6x + 32$	☐ Yes ☐ No	

The holistic rubric for scoring all three cases is as follows:

3 points – Correctly select “Yes/No,” and give right reasoning
2 points – Correctly select “Yes/No,” and give reasoning that has a minor flaw or needs clarification
1 point – Correctly select “Yes/No,” but the reasoning has a major flaw
0 points – Wrong choice, or give an answer that is wrong or without reasoning, or no answer

For each of the three pairs of equations, Table 5.5 shows the frequencies and percentages of answers with different scores:

Table 5.5 Summary of responses to Question 5.4

Score \ Pair	1)		2)		3)	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
3	47	65.3	21	29.2	28	38.9
2	9	12.5	14	19.4	9	12.5
1	4	5.6	8	11.1	5	6.9
0	12	16.7	29	40.3	30	41.7
Total	72	100.0	72	100.0	72	100.0

Only 9 participants (12.5%) gave fully correct answers to all three questions.

For equation pair 1), among the 47 answers that were given three points, 19 (39.7%) of them point to the fact that the difference between the two expressions on each side was the same (constant 3). This type of reasoning does link to the balancing method. The other 28 (59.6%) answers combined the constant terms within each of the two equations, both of which became $3x = 20$ or $3x - 20 = 0$.

Answers that have partial problems include, "The two equations have the same slope and *y*-intercept," "the two lines are identical," "the *x* coefficients are the same," and "they have the same slope." And answers that got zero points are those stating, "No, they are not equivalent" and giving reasons such as, "the second equation is not a scalar multiple of the first one," "their constants are different," or "I have to solve."

For equation pair 2), among the 20 answers that were assigned three points, 12 (60%) of them pointed to the fact that the difference between the two equations was not the same on the two sides. Again, this type of reasoning does link to the balancing method. The rest of the eight (40%) answers combined the like terms in the two equations, one of which became $2x = 23$, while the other was $x = 23$.

Answers that have minor or major problems include, “The two equations have the same *y*-intercept but different slope,” “there is not enough difference in *x* to match the difference in integers,” and “the *x* terms are not the same.”

And answers that got zero points are those stating, “No, they are not equivalent” and those giving reasons such as, “They have different slopes,” “their constants are the same,” “*x* is the same on one side but not the other,” “the two equations are not scalar multiple (or proportional),” and “I have to solve.”

For equation pair 3), all of the 28 answers that were given three points utilize the fact that all corresponding terms between the two equations have the same ratio except the terms seven and eight (i.e., there will not exist a single basic operation that simultaneously links the two corresponding sides of the equations).

Two examples of answers that got partial scores are, “No, the two lines have different *x*-intercepts” and “no, the two lines are parallel but not identical.”

Answers that scored zero points are those stating, “No, they are not equivalent” because “they have different slopes,” “their constants are the same,” “their constants are not the same,” “the *x* terms are not the same (or not balanced),” or “I have to solve.”

Two patterns in teachers’ knowledge and reasoning were observed from some of those incorrect or problematic answers: (a) Some teachers tend to examine the equivalence for all three cases in a single approach: either looking at whether the two corresponding sides have the same difference or looking at whether all the coefficients are proportional. (b) Some other teachers approached the problems in terms of *slope* and *intercept*, which suggests that they were used to thinking about equations from a geometric (or graphical) perspective, or, equations in two variables. More specifically, some of them consider two equivalent equations as those that have the same slope, y-intercept, or both. In an extreme case, when two linear equations (in two variables) have the same slope and y-intercept, their lines are coincident. This is consistent with my prior observations from the pilot studies where, rather than following the definition of equivalent equation that is provided, some teachers switched to a different concept with the same name: equivalent *forms* of an equation that represent the same curve, such as the slope-intercept form, the slope-point form, the two-point form, the standard form of the same linear equation, the vertex-axis of symmetry form, the two-point form, and the standard form of the same parabola.

Teachers' association of equivalent equations with the slope, the y-intercept, or both is further confirmed by the participants' responses to question 5.3, regarding the graphical features of equivalent linear equations, which are discussed below.

2. Equivalent equations and their graphical features

A linear equation $ax + b = 0$ has the following basic property: its solution is $x = x_0$ if and only if $(x_0, 0)$ is the x-intercept of the line $y = ax + b$. Following this property, assessment question 5.3 asks about the graphical features of two equivalent linear equations:

5.3 If $ax + b = 0$ and $cx + d = 0$ are two **equivalent** but **different** linear equations, what can we say about the two corresponding lines $y = ax + b$ and $y = cx + d$? Please determine the truth of each of the following statements:

	Always true	Possibly True	Impossible
1) These two lines are identical	☐	☐	☐
2) These two lines are parallel	☐	☐	☐
3) These two lines have the same *x*-intercept	☐	☐	☐
4) These two lines have the same *y*-intercept	☐	☐	☐
5) These two lines are perpendicular	☐	☐	☐

The frequencies and percentages of the participants' responses to each question are summarized in Table 5.6. The bold numbers correspond to the correct answers.

Table 5.6 Summary of responses to Question 5.3

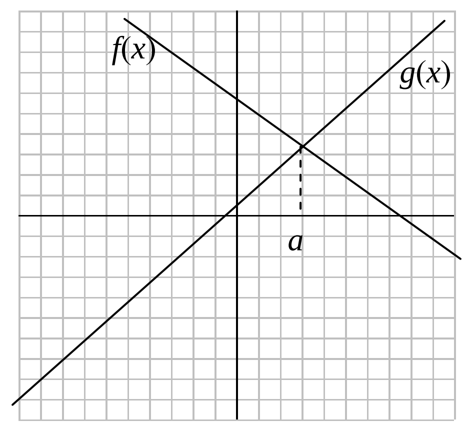
Statement / Choice	1)		2)		3)		4)		5)	
	n	%	*n*	%	*n*	%	*n*	%	*n*	%
Always	12	16.7	13	18.1	**35**	**48.6**	15	20.8	0	0.0
Possibly	30	41.7	20	27.8	24	33.3	**30**	**41.7**	**37**	**51.4**
Impossible	**28**	**38.9**	**36**	**50.0**	12	16.7	25	34.7	33	45.8
No answer	2	2.8	3	4.2	1	1.4	2	2.8	2	2.8
Total	72	100.0	72	100.0	72	100.0	72	100.0	72	100.0

None of the participants answered all questions correctly. Seventeen (23.6%) of the participants made four correct choices, 12 (16.7%) made three correct choices.

3. The balancing method and the intersecting-the-lines method

As discussed in the textbook review in Chapter 2, in solving a linear equation $f(x) = g(x)$ where both $f(x)$ and $g(x)$ are linear expressions, we can graph two linear functions $y = f(x)$ and $y = g(x)$, then determine their intersection (a, b). The x-coordinate of the intersection, a, is the solution to the original equation. This function-based method potentially provides a graphical representation of the balancing method and the corresponding solving process: What happens graphically whenever we apply a transformation on both sides of a linear equation? In the assessment, after the function-based approach was introduced, question 6.5 was asked to reveal teachers' understanding:

6.5 $f(x)$ and $g(x)$ are two given linear functions. The figure shows the function-based method for solving the equation $f(x) = g(x)$. Suppose the solution is $x = a$.



1) If we use the same method to solve a related equation $f(x) + 3 = g(x) + 3$, what would the graphs look like? Please sketch them in Figure 2.

2) Suppose $x = b$ is the solution to $f(x) + 3 = g(x) + 3$, what would be the relationship between a and b?

A. $a > b$ **B**. $a = b$ **C**. $a < b$ **D**. It depends on what $f(x)$ and $g(x)$ are

3) Further, if $x = c$ is the solution to another related equation, $3f(x) = 3g(x)$, what would be the relationship between a and c?

A. $a > c$ **B**. $a = c$ **C**. $a < c$ **D**. It depends on what $f(x)$ and $g(x)$ are

For question 1), 60 (83.3%) of the participants were able to sketch the graph correctly, which suggests they did understand of the fact that, graphically, $f(x) + a$ is the vertical shift of $f(x)$ by a units.

For questions 2) and 3), the numbers and percentages of participants who made a particular choice are shown in Table 5.7. In both questions, the correct choice is B. The corresponding numbers are bold:

Table 5.7 Summary of responses to Question 6.5 2) and 3)

	Question 2)		Question 3)	
Choice	Frequency	Percent	Frequency	Percent
A	2	2.8	10	13.9
B	**50**	**69.4**	**35**	**48.6**
C	14	19.4	9	12.5
D	4	5.5	15	20.8
No answer	2	2.8	3	4.2
Total	72	100.0	72	100.0

For question 2) regarding $f(x) + 3 = g(x) + 3$, even though the participants who made the correct choice were still the majority, the number of correct choices was much less than that from question 1). Specifically, 14 (23.3%) of the 60 participants who correctly sketched $f(x) + 3$ and $g(x) + 3$ did not figure out the correct relationship between a and b (choice B). Actually, 11 participants thought that a was less than b (choice C). Based on their responses to the assessment as well as the pilot study, these participants seemed to have mistakenly compared the y-coordinates of the two intersection points, rather than their x-coordinates. Or, they may have been distracted by the vertical translation and may not have focused on the real issue, the x-coordinate of the intersections. Figure 5.7 is a sample response from one of the participants who seems to have confused the intersection with the solution:

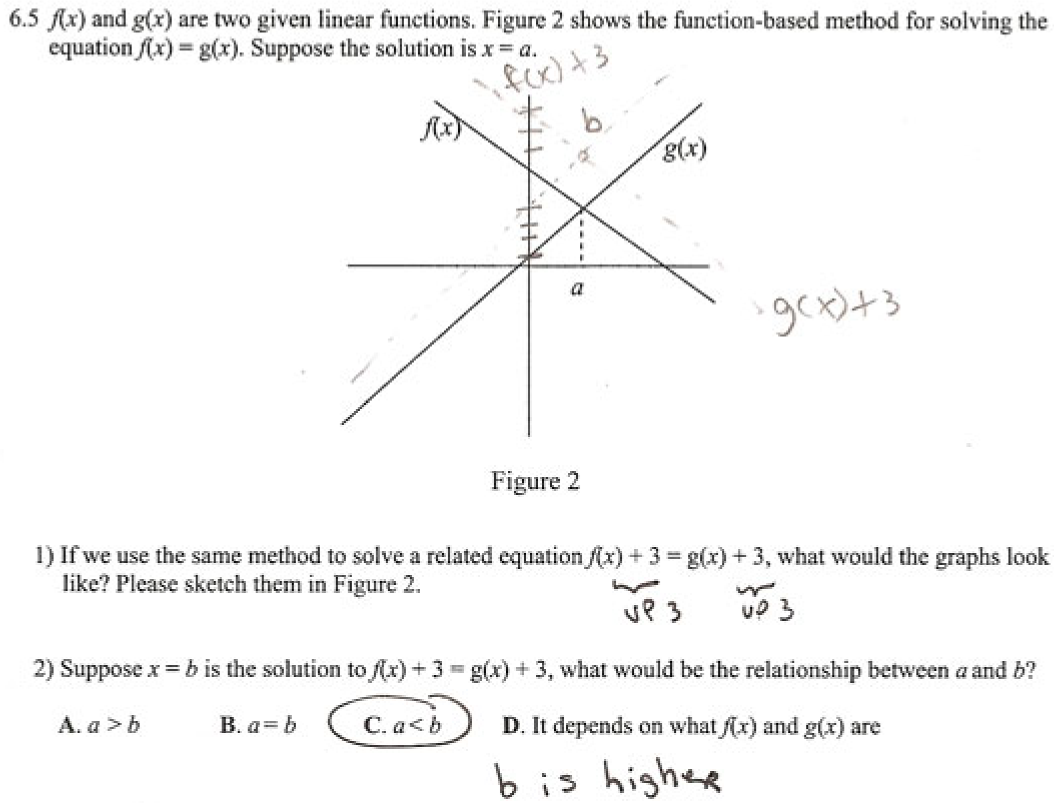

6.5 $f(x)$ and $g(x)$ are two given linear functions. Figure 2 shows the function-based method for solving the equation $f(x) = g(x)$. Suppose the solution is $x = a$.

Figure 2

1) If we use the same method to solve a related equation $f(x) + 3 = g(x) + 3$, what would the graphs look like? Please sketch them in Figure 2.

up 3 up 3

2) Suppose $x = b$ is the solution to $f(x) + 3 = g(x) + 3$, what would be the relationship between a and b?

A. $a > b$ **B.** $a = b$ **C.** $a < b$ **D.** It depends on what $f(x)$ and $g(x)$ are

b is higher

Figure 5.7 A sample response to Question 6.5 1) and 2)

Moving into question 3), which is about $3f(x) = 3g(x)$, fewer participants made the correct choice, and the numbers of participants selecting choices A, C, and, particularly, D, increased. My hypothesis is that the participants had a greater uncertainty in terms of how exactly the new graph $3f(x) = 3g(x)$ would look or where exactly the new intersection would be located.

To verify my hypothesis and gain insights into the knowledge and reasoning teachers use in solving this type of problem, I repeated questions 2) and 3) in the follow-up interviews and asked the participants to explain why they made certain choices. As an extension, I also added a potentially more complicated case in the interviews:

> If $x = d$ is the solution to the equation $f(x) + 3x = g(x) + 3x$, what would be the relationship between d and a? Why?

For $f(x) + 3 = g(x) + 3$, seven of the interviewees correctly answered that $b = a$ and gave three types of reasoning:

1. Graphical approach (Jane, Pam, Teresa, and Tom). They realized that, graphically, $f(x) + 3$ is a vertical shift of $f(x)$ up by three units and so is the relationship between $g(x) + 3$ and $g(x)$. Therefore, the intersection is up vertically by three units, while the x-coordinate does not change.

2. Using property of equality. Yvonne gave a quick justification: "The solutions are the same because you've just added 3 on both sides. You still *maintain the equality*." Earlier in the interview, Yvonne mentioned that instead of explicitly teaching her students the four properties of equality, she has emphasized the need to "maintain the equality," (i.e., always perform balanced operations on both sides).

3. Combination of graphical reasoning and symbolic justification (Amy and Mary). From the sketches they both believed the x-value of the intersections should be the same, and then they turned to symbolic processes for conviction. Amy used a specific equation as an example:

> If I have something like $x + 2 = 8$, I know x equals 6. And if I add 3 to this side and add 3 to this side…and solve the new equation, then that effect of adding 3 to each side doesn't change my x value because 6 is still making the equation true. So I would say that translating it up or solving this related equation would not change the x value of the point of intersection.

Mary subtracted 3 from both sides of $f(x) + 3 = g(x) + 3$ and got the original equation $f(x) = g(x)$, so she was convinced the solution should be the same.

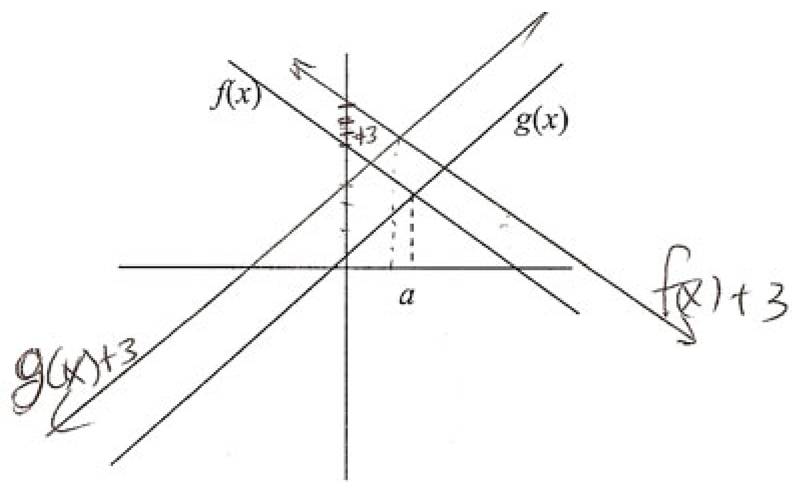


Figure 5.8 Rene's response to Question 6.5 1)

Rene was the only one who gave an incorrect answer: $b < a$. Her sketch (see Figure 5.8) revealed the cause: she understood that both lines go up by three units but did not realize the intersection has only a vertical shift. The horizontal shift toward the left led to her answer of $b < a$. Obviously, she also understood the meaning of the solution: it is the x-coordinate, not the intersection itself.

The situation got more complicated for the question regarding $3f(x) = 3g(x)$. All of the participants tried two or more of the following strategies:

1. Using the properties of equality. Tom divided both sides by 3 and derived $f(x) = g(x)$, so he immediately drew the conclusion that the solution was the same. Yvonne

stated, “It shouldn’t matter ‘cause you can just do cancelling.” Pam realized the new equation could be viewed as the same thing (multiplying by 3) being done on both sides, so the solutions were the same. They all believed the sketches would lead to the same conclusion but were not able to validate such hypothesis in the graphs.

Amy first thought multiplication was repeated addition, so she could undo multiplication through undoing addition. Then she claimed, “All I’m doing is either adding, or multiplying…either adding a number, not a variable, or multiplying by a number, not a variable. It’s not gonna change the value of the variable that makes the original statement true.” This actually led to her later confusion in applying this strategy to $f(x) + 3x = g(x) + 3x$.

2. Sketching the lines. Like they did for $f(x) + 3 = g(x) + 3$, several of the participants first attempted to sketch $3f(x)$ and $3g(x)$. However, they only paid attention to the change in slope, without being sensitive to whether and how the intercepts would change. Consequently, their sketches were all skewed (as an example, Figure 5.9 shows the sketch by Rene). Feeling uncertain about or unconvinced by the conclusion $c < a$, some of them turned to other strategies.

3. Manipulating general equations. Rene also tried to write $f(x)$ in the general form $mx + b$, then multiply it by 3 but was not able to make further progress (see Figure 5.9).

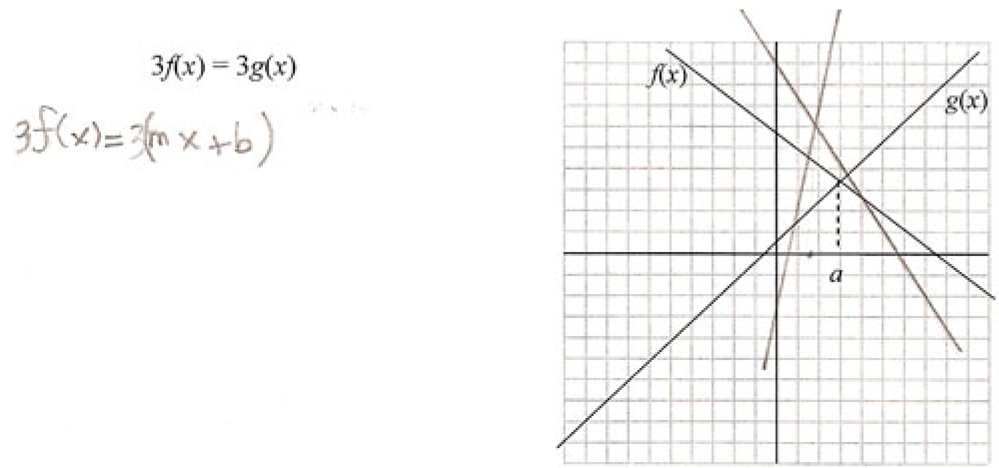


Figure 5.9 Rene's response to Question 6.5 2)

4. Calculating the parameters. Teresa started by saying, "It's just hard without knowing the slope." So she analyzed the graphs of $f(x)$ and $g(x)$ as if they were two specific functions. She figured the slope of $f(x)$ was about $-\frac{3}{4}$ and then multiplied it by 3 to get the slope of $3f(x)$, $-\frac{9}{4}$ (see Figure 5.10).

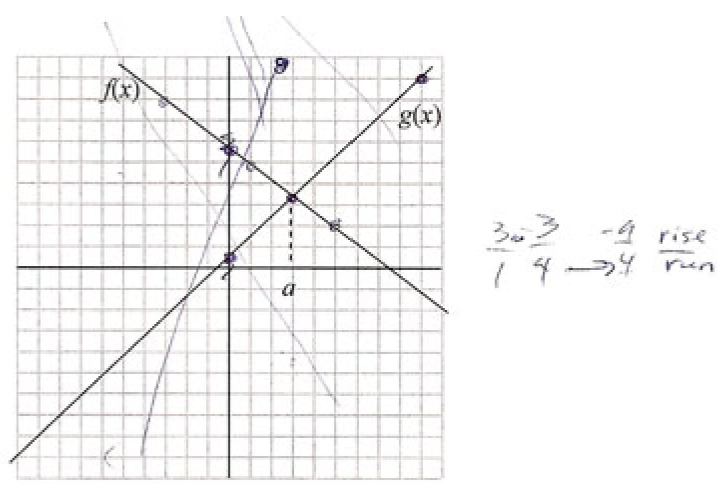


Figure 5.10 Teresa's response to Question 6.5 2)

But Teresa could not make further progress beyond the realization that both lines are “gonna be higher and slanter *[sic]*,” or “sharper and up.” She was not sure why she chose $a = c$ in the written-assessment.

5. Experimenting with special cases. When they realized that sketching based on the two lines provided would not give them a definite answer, Jane and Mary turned to special cases. Jane used two lines that went across the origin (see Figure 5.11) and quickly drew the conclusion that the solution stayed the same. When asked whether this would still be true if the lines were not crossing the origin, she pondered this and eventually turned to the properties of equality for conviction.

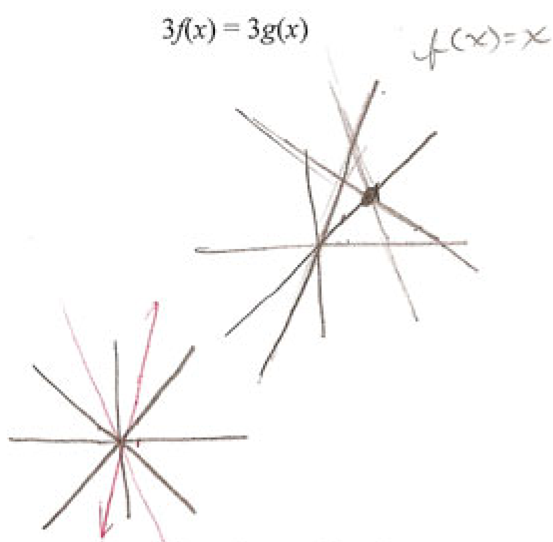


Figure 5.11 Jane’s response to Question 6.5 2)

Mary (see Figure 5.12) first tried $2x + 1$ for $f(x)$ but quickly gave up. Then, she used $f(x) = x$ and $g(x) = -x$ as the new cases and made numerical tables for $3f(x)$ and $3g(x)$, which turned out not to be helpful either.

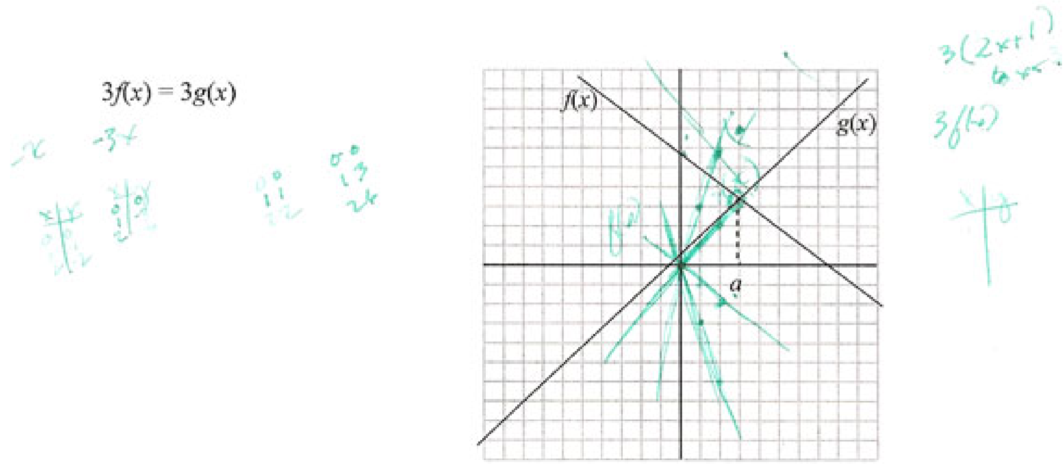


Figure 5.12 Mary's response to Question 6.5 2)

The participants' responses to the question about $f(x) + 3x = g(x) + 3x$ were basically an extension of their thinking on $3f(x) = 3g(x)$. Jane, Pam, Tom, and Yvonne continued to use the addition property of equality to draw the quick conclusion that $d = a$. Teresa started with a graph but switched to the addition property for justification. Mary and Rene were stuck analyzing the sketches, numerical parameters, and data, so they still could not figure out the final convincing arguments. Amy was the only exception. She was not sure whether the solution would stay the same when she saw that a term containing the variable x was added to both sides.

Teachers’ Responses to Contextualized Items

When contextual information about teaching and learning (e.g., responding to a specific idea from students, representing a mathematical concept or process with certain manipulatives, helping students improve their understanding of a certain concept or process) is introduced into a problem situation, teachers may need to go beyond subject matter knowledge and utilize the other two forms of mathematical knowledge for teaching: knowledge of learners’ conceptions and knowledge of didactic representations. This is when teachers’ various preferences come into play.

Descriptive statistics

On those contextualized assessment questions, the participants’ subtotal scores ranged from 5 to 35 points (out of a full score of 38 points), with mean score $\mu = 19.8$, median $\mu_{1/2} = 20.5$, standard deviation $\sigma = 7.2$, and standard error mean $S_E = 0.9$. Figure 5.13 is the histogram of the subtotal scores:

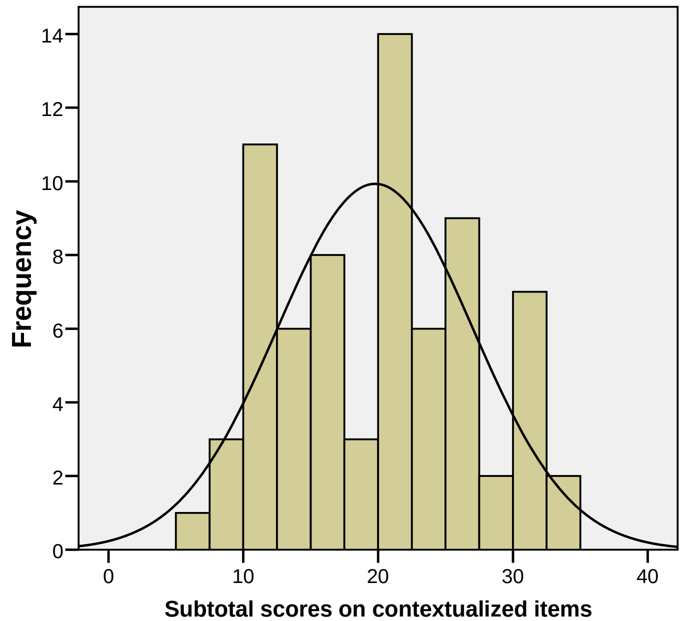


Figure 5.13 Histogram of subtotal scores on contextualized problems

Identifying typical student mistakes or difficulties

1. The balancing method

Question 1.2 in the assessment asks the participants to identify two different types of mistakes or difficulties that they have seen from their students in learning and using the balancing method. Altogether, the 72 participants provided 138 cases of mistakes and difficulties, which covered a wide spectrum and were categorized into seven groups by their mathematical nature plus one catch-all group. The number of cases in each group and the corresponding percentages are shown in Table 5.8:

Table 5.8 Summary of responses to Question 1.2 Part 1

Categories of student difficulties and mistakes	Frequency	Percentage
1. Negative sign, subtraction, additive inverse	47	34.1
2. Imbalanced operations on the two sides	19	13.8
3. Combining unlike terms	15	10.9
4. Multiplicative inverse	13	9.4
5. Misuse of the distributive property	13	9.4
6. Order of operations	10	7.2
7. Fractional coefficients	9	6.5
8. Other types	12	8.7
Total	138	100

The first category of mistakes and difficulties has to do with students' understanding of three related concepts and procedures: the negative sign, the subtraction operation, and additive inverses. Over one-third of the cases provided by the participants fall into this category, which makes them the most frequent among all mistakes and difficulties. This also echoes what one of the participants noted in her answer: This type of mistake is the "number one problem in high school algebra." Such mistakes and difficulties have their roots in elementary school arithmetic and have been documented and studied in research literature, as shown in Chapter 2.

Specifically, the first category of mistakes and difficulties includes the following major cases:

1. Do not understand the connection between negative sign and subtraction ($a - b = a + (-b)$, $a - (-b) = a + b$)

2. Problems with negative sign (e.g., in solving $2 - 3x = -13$, after subtracting 2 from both sides, a student writes down $3x$ rather than $-3x$), or, always wanting to change the sign of a variable even if it stays on the same side

3. Do not do the right inverse operation, particularly when the equation is not in $ax \pm b = c$ form (e.g., adding 9 on both sides when solving $9 - 2x = 3$, which could be because the student mistakenly recognizes the subtraction following 9 as the negative sign for 9)

The second category, imbalanced operations on the two sides, includes two major cases:

1. Do an operation on both terms on one side, but not on the other side. For example,

$$\begin{array}{l} 4x - 2 = 5 \\ \underline{+2 \;\; +2 \qquad} \end{array}$$

2. Do one operation on one side, but do the inverse on the other side:

$$\begin{array}{l} 4x - 2 = 5 \\ \underline{\qquad +2 \;\; -2 \;\;} \end{array}$$

As reviewed in Chapter 2, this case above is documented in Kieren (1984), who names it the *redistribution error*. In one of the follow-up interviews, Yvonne also brought up this case but could not exactly explain the cause.

The other categories of mistakes and difficulties are either related to arithmetic knowledge and skills (problems with multiplicative inverse, order of operations, and dealing with fractional coefficients) or arising as new issues when students begin to learn algebra (combining unlike terms and do not follow the distributive property).

In the order of operation category, several participants listed "dividing and multiplying before adding and subtracting" (or "don't know to clear numbers first") and "begin with the wrong side" as student mistakes. These indicate that some teachers may overemphasize certain routines or rules which may not necessarily have to be followed in all circumstances. In the follow-up interviews, however, all teachers understood that students should not have to follow such rules all the time. For instance, one could absolutely divide the two sides first for an equation such as $3x + 9 = 6x - 15$, or multiply the two sides first when one or some of the coefficents are fractions.

For each mistake or learning difficulty they mentioned, the participants were also asked to provide strategies for helping students to improve their understanding. The strategies are discussed in the next section, knowledge of didactic representations.

2. The quadratic formula

A similar question is also asked in the assessment (question 4.3) about the typical student mistakes and difficulties in applying the quadratic formula. A total of 128 cases were provided by the participants. Table 5.9 shows the major categories, the number of cases in each category, and the percentage out of the total:

Table 5.9 Summary of responses to Question 4.3 Part 1

Categories of student difficulties and mistakes	Frequency	Percentage
1. Computations under the square root, order of operations	34	26.6
2. Negative signs and rules for integer operations	25	19.5
3. Dealing with the $\pm$ symbol	23	18.0
4. Identifying the coefficients when the equation is not in standard form	15	11.7
5. Dividing by $2a$	14	10.9
6. Forgetting the formula	11	8.6
7. Computational errors	6	4.7
Total	128	100

Having multiple operations and the new symbol ($\pm$) integrated in one formula, it is not surprising that the quadratic formula could cause all kinds of errors when it is being used by algebra students.

The first category, computations under the square root, involves mistakes and difficulties that originated from arithmetic (order of operations with $b^2 - 4ac$) as well as those associated with the relatively new and complex operations: square root (simplifying the square root after $b^2 - 4ac$ is computed).

The second largest category of mistakes and difficulties has to do with the handling of the negative sign in various situations: mistaking the signs of a and c when they are negative, dropping the negative sign for b which is in front of the square root, and adding a negative sign for b^2 when b is negative.

The strategies that are provided by the participants will also be discussed in the next section, knowledge of didactic representations.

Analyzing and responding to student thinking

Teachers' analysis of student thinking is, first of all, a matter of determining the mathematical rationales of the students' claims or questions. Beyond that, it may require teachers' familiarity with students' thinking habits. Questions that ask how to respond to students may elicit diverging types of answers.

1. Understanding the nature of a method

Question 6.2 probes the participants' understanding of the nature of the intersecting-the-lines method. Table 5.10 shows the summary of their responses:

> 6.2 Danny believes that the above method works for a linear equation only when the unknown variable x appears on both sides, i.e., it won't work if one side of the equation is a constant (for instance, $2x + 9 = 8$, or $5x - 13 = 0$).
>
> What would you say to him?

Table 5.10 Summary of responses to Question 6.2

Score	Frequency	Percent
3	31	43.1
2	9	12.5
1	8	11.1
0	24	33.3
Total	72	100.0

Problems with some of the responses include mistakenly saying that "the constants are vertical lines"; stating that "the method still works" without clarifying why; simply telling the student "it works, try it," "use your graphing calculator to check," or "move the constant to the other side so it becomes zero"; or claiming that the method does not work on these cases.

2. Fundamental difference between similar concepts and processes

There are some things in common between solving a linear equation $ax + b = cx + d$ through the function-based approach and finding the graphical solution to a related system of linear equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$: They both involve graphing the two lines $y = ax + b$ and $y = cx + d$ and then finding the intersection (x_0, y_0). But a subtle difference between these two processes is in the dimensions of the final solutions: The solution to the system is the point in the x-y plane, or the ordered pair (x_0, y_0), while the solution to the linear equation is a number (a point on the real number line) or the x-coordinate of the point (x_0, y_0), x_0. Below is a question in the assessment:

> 6.4 Emily has learned how to use the above method to solve linear equations like $3x + 5 = -4x - 2$. Now she's learning the graphical method for solving linear systems such as $\begin{cases} y = 3x + 5 \\ y = -4x - 2 \end{cases}$: graph the two lines $y = 3x + 5$ and $y = -4x - 2$, then find the intersection. She is happy to find out the connection: "These two methods are actually the same and they give the same solutions!"
>
> Is her conclusion valid? What would you say to her?

Table 5.11 shows the frequencies and percentages of different scores:

Table 5.11 Summary of responses to Question 6.4

Score	Frequency	Percent
3	24	33.3
2	8	11.1
1	5	6.9
0	35	48.6
Total	72	100.0

Exactly one-third of the answers got full scores (three points), which means the teachers are clear about the difference between the two solutions.

Here are some responses that are more or less vague or problematic (two points or one point): "Methods are the same but solutions are different." "The first has one variable, the second has two." "Yes it's valid. Graphing leads to (x, y), whereas algebraic method gives x-coordinate only."

Responses that indicated that the two methods and solutions were the same scored zero points (for instance: "That is correct. There are multiple ways to find the same solution." "Yes it's like substitution, but backwards, or undone!" "Yes. If $3x + 5 = -4x - 2$ and then both sides are set equal to the same thing, $y = 3x + 5$ and $y = -4x - 2$, it makes sense that you get the same solution.").

The distributions of the participants' sum scores on Questions 6.2 and 6.4 are shown in Table 5.12 and Figure 5.14, and comparison is made based on whether they have taught the method before:

Table 5.12 Sum scores on Questions 6.2 and 6.4 by teaching experiences

Participants / Sum score On 6.2 and 6.4	Have taught the method		Have not taught the method	
	Frequency	Percentage	Frequency	Percentage
5 or 6 points	14	41.2	8	21.1
3 or 4 points	9	26.5	13	34.2
1 or 2 points	3	8.8	5	13.2
0 points	8	23.5	12	31.6
Total	34	100	38	100

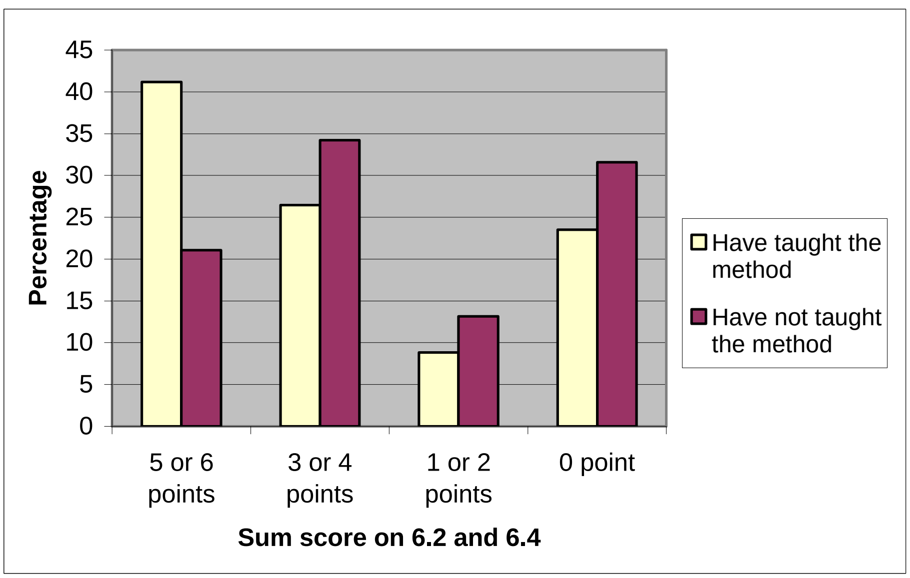


Figure 5.14 Distribution of scores on question 2.3 by participants' experiences

The distributions of sum scores show that the percentage of teachers who have (a) taught this method and (b) scored high on these two questions is much higher than those who have not taught it, but, beyond that, we would not know if teaching experience has a significant impact on teachers' knowledge. On the one hand, 21.1% of the teachers who have not taught the method before still scored very high; on the other hand, nearly one-third of those who have taught the method scored very low on the two questions. These seem to support the hypothesis that teachers' subject matter knowledge plays a key role in answering these types of questions.

3. Overgeneralization of properties or methods

Going beyond identifying student mistakes and difficulties, sometimes mathematics teachers would also need to figure out what may cause a particular mistake or difficulty. Question 3.6 of the assessment begins with probing teachers' knowledge of a student's thinking in solving a quadratic equation:

3.6 Below is how Mark solved the quadratic equation $x^2 - 5x - 1 = 0$:

$$x^2 - 5x = 1$$
$$x(x - 5) = 1$$
$$x = 1 \text{ or } x - 5 = 1$$
$$x = 1 \text{ or } x = 6$$

1) What might he be thinking when he decided to solve the equation in that way?

Twenty-five (34.7%) of the participants believed that Mark was either trying to apply the zero product property as if the similar property were true, even when the right-hand side of the equation was 1, or that he was misapplying strategies for $ax^2 + bx = 0$ type of equations. These explanations are considered the most reasonable ones. Fifteen (20.8%) of the participants thought that Mark was trying to isolate the variable or move all variable terms to one side. And six participants (8.3%) claim that Mark was moving the constant to the right-hand side, so that it would become easier to factor the terms on the left. These two types of answers touch part, but not all, of what happened in Mark's solution. For example, when $mn = 1$, Mark seemed to believe that $m = 1$ or $n = 1$.

Another 17 participants (23.6%) gave other kinds of explanations that were considered vague or incomplete to a certain degree, or that made little sense. For example: "He was confusing the identity vs. zero for multiplication," "He was trying to get factors," "The original equation doesn't factor, but the second one does," "He thought you can factor and set each equal to any number," "He was using the reverse of the distributive property to remove a monomial," "He forgot to set the equation to 0," and "He was trying to complete the square."

The second half of question 3.6 asks how the participants would response to the student:

> 2) Mark doesn't understand why this method won't work. How would you explain to him?

Twelve (16.7%) of the participants' responses go straightly to the key issue: $ab = 1$ does not imply $a = 1$ or $b = 1$. Many of them showed examples or asked the student to come up with his or her own examples. Seven (9.7%) of the participants wanted to review the zero product property ($ab = 0$ if and only if $a = 0$ and $b = 0$), show examples, or emphasize further that only zero has such property. Another 12 participants (16.7%) stated both of the above facts in their responses.

Among the 25 participants who indicated in their answers to the first question that Mark was over-generalizing the zero product property, 18 (72%) provided one of the three kinds of answers listed above. We may make the inference that when a mathematics teacher is able to see the real nature of the problems in student thinking, he or she is more likely to provide substantial feedback to the student.

Twenty-one participants (29.2%) responded that the equation has to be in $f(x) = 0$ form in order to be solved with the factoring method. Although this is true, it still does not fully explain the nature of the zero product property or, particularly, the fact that such property does not apply to 1.

Among the rest of the responses, some simply pointed out the mistake without providing hints for making improvement: “It is impossible for $x = 1$ and $x – 5 = 1$ to be true at the same time” or “neither 1 or 6 is correct.” Others were even farther away from being relevant: “Graph: it doesn’t cross the x-coordinates, finding solution is the same as finding x-intercept,” or “Need to do opposite operation.”

Representing mathematics processes with manipulatives

Question 1.3 of the written assessment asks the participants to describe their familiarity with two popularly used manipulatives for teaching and learning equation solving: balance scales and algebra tiles. The results are summarized in Table 5.13:

Table 5.13 Summary of responses to Question 1.3

Familiarity	Balance scale		Algebra tiles	
	Frequency	Percent	Frequency	Percent
I have taught with this model	31	43.1	53	73.6
I have seen or read about it but have never taught with it	35	48.6	14	19.4
I know little about it	6	8.3	5	6.9
Total	72	100.0	72	100.0

A few question were then asked about the use of these manipulatives.

1. Using balancing scales to solve linear equations

Question 1.4 is about the characteristics of balance scales in representing linear equation solving processes:

> 1.4. Is it possible to solve the equation $2x + 1 = 5x + 7$ by drawing pictures of weights and balance scales?
>
> If yes, please demonstrate how it can be done. If not, please explain why not.

The solution to the equation is $x = -2$, so the key issue behind this question is whether we can represent negative weights. Forty-four participants (61.1%) demonstrated the solving process by automatically introducing negative weights (see Figure 5.15 shows an example of the participants' solutions).

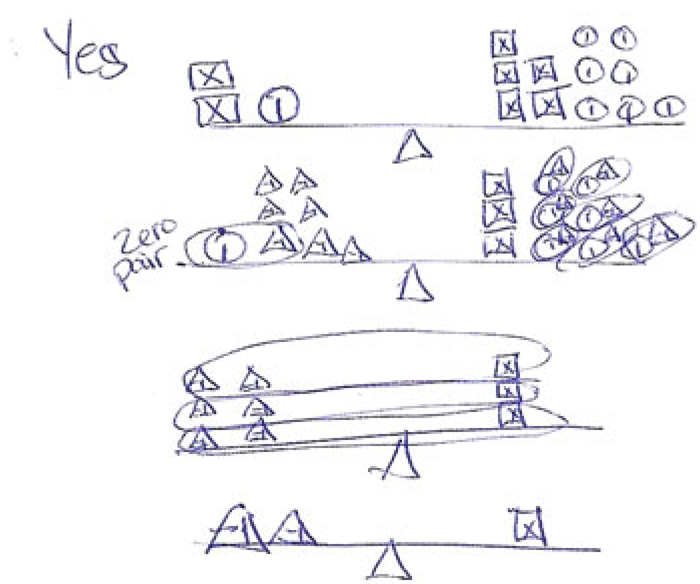


Figure 5.15 A sample response to Question 1.4

Two of the 44 participants illustrated special strategies for counterbalancing the positive weights (a) with a balloon (see Figure 5.16) or (b) by stacking blocks underneath (see Figure 5.17):

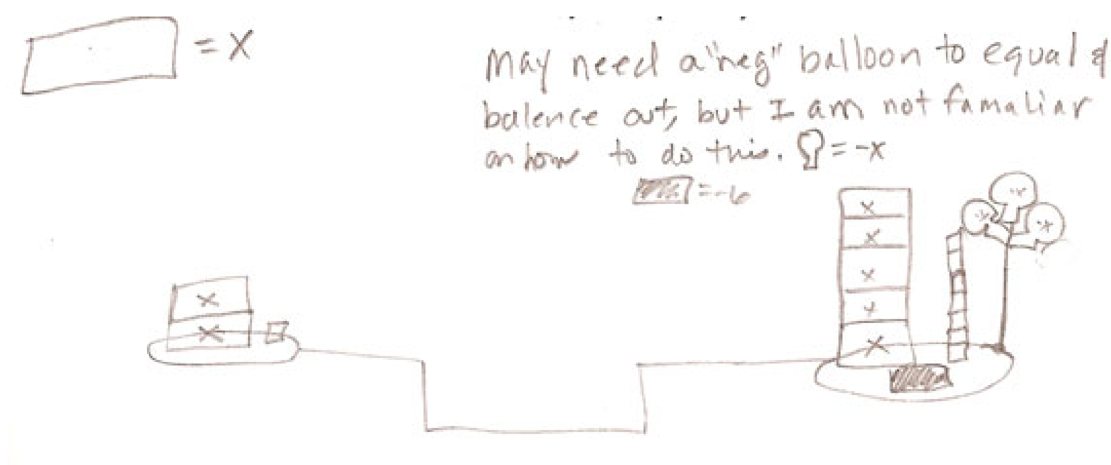


Figure 5.16 A response to Question 1.4 using balloons

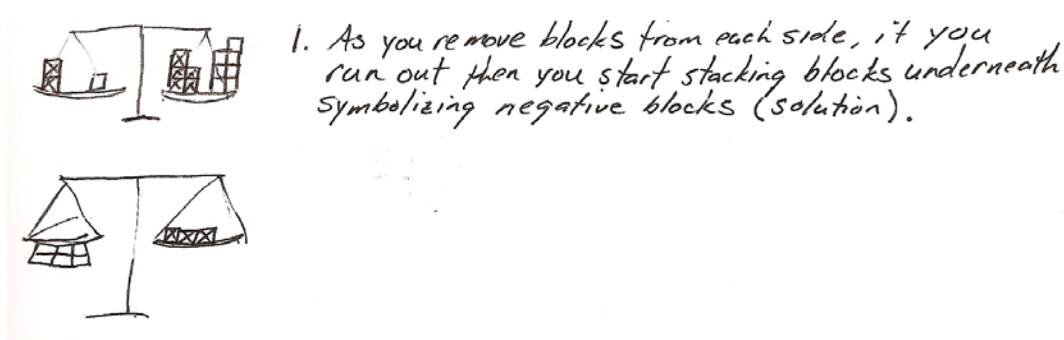


Figure 5.17 A response to Question 1.4 using underneath blocks

Thirteen (18.1%) participants gave negative responses. Most of them believed that there could not be negative weights, while two participants argued that after $2x$ and 1 were removed from both sides, there was nothing left on the left-hand side, so the scale became imbalanced (see Figure 5.18).

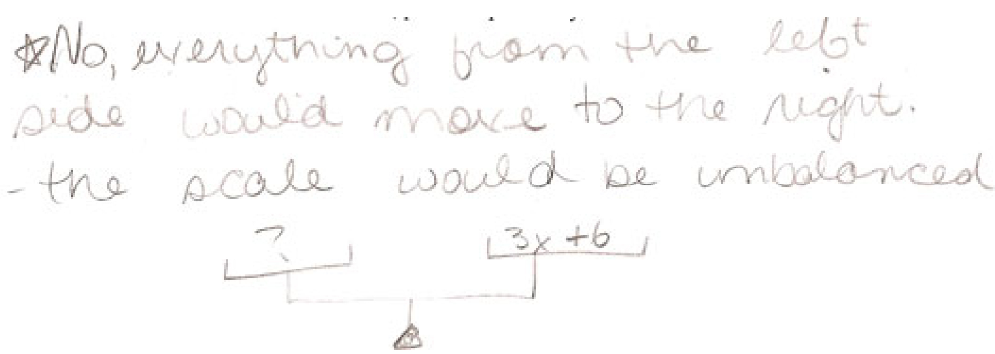


Figure 5.18 A response to Question 1.4 about imbalance

One participant explained that, mathematically, it is acceptable to talk about negative weights but, scientifically, it is not:

> The students loose *[sic]* the concept of balance scientifically if the balance was used only to show equality. It does not make sense to use a form that depicts something mathematically that is not true scientifically. In other words, a positive x does not weigh more than a negative x. Even if you decide to give each number a weight, what weight would that be? How could you prove the weight is the valid volume for any real number? Using a balance does not do math or science justice.

The rest of the 13 (18.1%) participants gave incomplete or no answers.

2. Using algebra tiles to solve linear equations

Question 1.5 probes whether teachers are sensitive to the potential limitations of using the algebra tiles to solve linear equations:

1.5 For each of the following equations, is it possible to use algebra tiles to demonstrate the solving process and accurately represent the solution? If not, please explain why.

	Yes	No	Explain why if you choose "No"
1) $5x + 3 = 14$	☐	☐	
2) $-3x + 5 = -10$	☐	☐	
3) $4x + 9 = 2x - 3$	☐	☐	

Table 5.14 shows the number and percentages of participants that made certain choices for each equation above. The correct answers are bold.

Table 5.14 Summary of responses to Question 1.5

Case / Choice	1)		2)		3)	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
Yes	30	41.7	**68**	**94.4**	**65**	**90.3**
No	**42**	**58.3**	4	5.6	7	9.7
Total	72	100.0	72	100.0	72	100.0

Question 1) assesses whether the participants can solve the equation correctly and realize that the solution ($x = 5/11$) cannot be represented by algebra tiles. Among the 42 participants who correctly selected "No," 35 (83.3%) have taught with algebra tiles and seven (16.7%) have never taught with them but have seen or read about them before. On the other hand, among the 30 participants who incorrectly selected "Yes," 25 (83.3%) have taught with or seen algebra tiles before and the other 5 (16.7%) have not.

Question 2) assesses whether the participants see an equation with negative coefficients as problematic when it is represented by algebra tiles. Among the 68 participants who correctly selected "Yes," 52 (76.5%) have taught with algebra tiles and 12 (17.6%) have never taught with them but have seen or read about them before. Only two of them know little about algebra tiles. On the other hand, among the four participants who incorrectly selected "No," three have taught with or seen algebra tiles before, and the rest have not.

Question 3) assesses whether the participants see an equation with a negative solution as problematic when it is represented by algebra tiles. Among the 65 participants who correctly selected "Yes," 50 (76.9%) have taught with algebra tiles and 13 (20.0%) have never taught with them but have seen or read about them before. Again, only two of them know little about algebra tiles; they are the same two participants who have little experience with algebra tiles but made the correct choice for question 1). On the other hand, among the seven participants who incorrectly selected "No," four have taught with or seen algebra tiles before, and the other three have not.

These above results seem to suggest that having certain prior experiences with algebra tiles (either having taught with or seen them) is a necessary, but not sufficient, condition. There are two participants who have little prior experiences with algebra tiles but made correct choices for both questions 2) and 3). It might be possible that they just guessed the answers without solid reasoning because they actually selected straightly "Yes" for all three questions.

3. Using algebra tiles to solve quadratic equations

Question 3.3 asks the participants whether they have taught their students how to use algebra tiles to factor a trinomial. Thirty-eight (52.8%) answered "Yes," and 33 (45.8%) answered "No." Question 3.4 then examines if they can actually factor trinomials with algebra tiles:

> 3.4 For each of the following two quadratic equations, is it possible to use algebra tiles (or draw their pictures) to solve it? If yes, please show how. If not, please elaborate why not.
>
> 1) $x^2 - 5x + 4 = 0$
>
> 2) $x^2 + 3x - 4 = 0$

Among the 38 participants who have taught their students factoring with algebra tiles, 30 (78.9%) were able to illustrate correctly the factoring results for both trinomials. Meanwhile, for those 33 participants who have not taught this before, 22 (66.7%) were completely wrong on both cases or gave no answer to either case. These seem to indicate a relatively strong connection between teachers' knowledge in using the algebra tiles to factor trinomials and their related teaching experiences.

Clarifying unconventional terms and issues

Mathematical concepts, facts, and reasoning are often highly sensitive to the constraints on the variables and systems involved. Specific to theories on polynomials and equations, there are major changes to the fundamental properties (such as reducibility of polynomials and solvability of equations) when the variables and coefficients are defined in different number systems. However, in mathematical communication, we may often make assumptions about certain conditions or overlook some implicit constraints. Assessment questions 3.2 and 3.5 touch upon teachers' knowledge of and sensitivities to factoring trinomials.

In general theories of polynomials, the domain for the coefficients of a polynomial must be specified before any conclusion about the reducibility (factorability) of the polynomial can be made. In secondary school algebra, however, the domain is typically not mentioned since it is assumed to be the set of all integers, ***Z***. An undefined term, such as "factorable trinomials," could work just fine under such assumption, but it might soon create contradictions once the quadratic formula was introduced after a short while: a "non-factorable" trinomial (on ***Z***) such as $x^2 + 4x + 1$ would suddenly become "factorable" (on the real number set, ***R***), and all trinomials would eventually become "factorable" (on the complex number set, ***C***). To maintain the longitudinal consistency, algebra teachers do need to be knowledgeable about the hidden information of domains.

Question 3.2 on the assessment was designed to test such knowledge:

> 3.2 An algebra teacher says the equation $x^2 + 4x + 6 = 0$ cannot be solved with the factoring method because the trinomial $x^2 + 4x + 6$ is "not factorable." Do you agree?
>
> If yes, how would you explain to your students what exactly it means that a trinomial is "not factorable"? If no, why not?

The majority (59, or 81.9%) of the participants agreed that the trinomial is not factorable. Among them, only six participants were able to explain clearly that it is not factorable when all the coefficients are limited to integers, while 44 participants did not address the issue of the domain at all (for example, by saying, "Not factorable means it cannot be written as the product of two or more polynomials," or "None of the pairs of factors of 6 add up to 4").

Seven participants (9.7%) believed that the trinomial was factorable, but their explanations were not all clear or reasonable. Five participants (6.9%) gave mixed responses, such as, "It is not factorable into real roots but factorable in Algebra 2 with imaginary"; "It is factorable. It doesn't factor in the sense of integer (or area model)"; or "It is not factorable by the factoring method, it is factorable by completing the square or the quadratic formula."

Question 3.5 involves a fundamental issue of polynomials: unique factorization. To a large extent, the problem has to do with the system in which the coefficients are defined:

> 3.5 In solving the equation $2x^2 - 5x - 3 = 0$, Karen factors it into $(2x + 1)(x - 3) = 0$. She then asks, "*Is this the only way of factoring it? How do we know?*"
>
> 1) How would you respond her?
>
> 2) Tony claims he did find a different way of factoring: $2(x + \frac{1}{2})(x - 3) = 0$. Is this valid? How would you respond to him?

For question 1), 42 participants (58.3%) believed Karen's factorization was unique. Among them, 17 (40.5%) explained that if we list all the factors of the leading coefficient 2 (1, -1, 2, -2) and those of the constant 3 (1, -1, 3, -3), the only possibility to make $-5x$ is to use 2 and 1 as the leading coefficients for the linear factors, and use 1 and -3 as the constants in the factorization. Other explanations included, "It is guaranteed by the Fundamental Theorem of Algebra," "It is the same as the unique prime factorization of integers," "There are two other solutions," and "There are other factors but we only take the ones with positive coefficients."

Twenty participants (27.8) did not think the factorization was unique. At least nine of them actually made statements like, "There are other methods (such as the quadratic formula) to do this," which indicates that they may have misinterpreted Karen's question as, "Is this the only way of finding the factors?" Others were mostly vague or inaccurate arguments, such as, "It's not unique because of the 2," or "Both $2x + 1$ and $x - 3$ are prime."

For question 2), 52 (72.2%) of the participants believed Tony's factorization was valid. Among them, 16 responded that the two ways of factoring were equivalent and the solutions were the same. Other responses included, "It is not necessary (or too complicated" to do in this way," "It makes it harder," "You can't factor 2 out otherwise you can't use the zero product property" (or "otherwise it becomes 2=0"), and "Using whole number factor is more beneficial for graphing." Only two participants seemed to be somehow aware of the coefficient issue: "Factoring is about integers," and "We use whole number factors only."

Teachers' Responses to More Open-ended Items

When the problem situations become more complicated, teachers may simultaneously draw upon more than one form of knowledge or have preferences over a certain form of knowledge.

Descriptive statistics

On those more open-ended questions, the participants' subtotal scores range from 3 to 20 points (out of a full score of 24 points), with mean score $\mu = 13.2$, median $\mu_{1/2} = 14.0$, standard deviation $\sigma = 3.4$, and standard error mean $S_E = 0.4$. Figure 5.19 is the histogram of the subtotal scores:

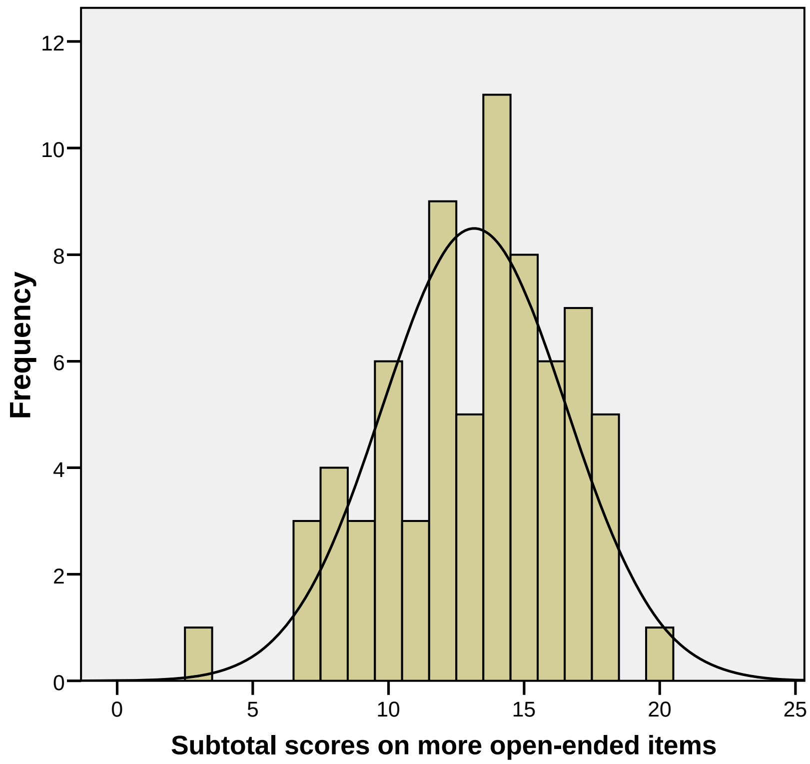


Figure 5.19 Histogram of subtotal scores on more open-ended items

Strengths and advantages of a certain method

1. The balancing method

After explaining the nature and the context of the balancing method, Question 1.1 asks about the strengths of this method:

1.1 Why is the balancing method the most commonly taught and used strategy for solving linear equations? Please give two reasons that you believe are the most important.

Below are the representative reasons provided by the participants. By *representative*, I mean they include all different reasons, and when two reasons are essentially the same, despite differences in wording, they are listed as one reason. They are put into five categories based on their major focuses. The first three categories align with the conceptual framework: reasons focusing on the mathematical subject matter, student understanding, and didactic representations. Another category emerged from one type of response: reasons focusing on traditions (in the teaching and learning of mathematics). All other reasons are labeled “Others.”

1. Reasons focusing on the mathematical subject matter

- It helps students to understand the meaning of equal and how to keep equal
- It is quick and precise
- The original problem is never changed, only the form is changed (concept of equivalence)
- The idea of inverse operations is important, it lays foundation for inverse function
- It reinforces order of operations
- It is logical
- It promotes the idea of symmetry
- It is a simple process that works consistently
- It’s an algebra method, does not require manipulatives
- It is important for more advanced study
- The equal sign represents a balance
- This method frees us from depending on manipulatives
- It is justified by the commutative, associative, and distributive properties
- It works on a variety of problems

2. Reasons focusing on pedagogical representations

- It is easy to demonstrate using manipulatives
- It is easy to teach without manipulatives
- It is easy to show a step-by-step process
- One can teach in steps like a formula

3. Reasons focusing on student understanding

- Many students are taught to think of the equal sign as a balance
- It builds on balance scale model and balanced ideas, which are familiar to students
- It is linked to a visual model (scale) that students can picture
- Students see and understand why they must do the same on both sides
- The steps are concrete and students can follow along
- The principle is easy for teachers to state and easy for students to remember and understand
- It provides a formula/routine for students to use
- Once students learn it, it is easy to repeat
- It can be memorized and performed easily by students
- It is easy for students to recognize inverse operations
- It is an easy mnemonic ("What I do to one side, I do to the other side")
- It relates equation solving to real life (scale as real world object)/something they already know
- It links abstract to concrete concepts
- It is easy to learn

4. Reasons focusing on traditions

- Tradition! It is what my Mamma did
- It is the way I learned it, I understand it well
- It has always been done that way, teachers are comfortable with it
- It is presented in most textbooks I used

5. Reasons with other focuses

- It is connected to the balancing idea in other disciplines

Table 5.15 displays the distribution of the five types of reasons:

Table 5.15 Summary of responses to Question 1.1

Emphasis	Frequency	Percentage
Learner conceptions	72	50.0
Subject matter	42	29.2
Didactic representations	12	8.3
Tradition	12	8.3
Other	6	4.2
Total	144	100.0

Eighteen participants (25%) had two answers both focusing on learner conceptions. Nine participants (12.5%) gave two reasons that are both focused on the subject matter. The rest participants gave two answers that are of mixed types.

2. The undoing method

After the undoing method was introduced and some basic questions were asked, question 2.4 attempted to uncover teachers' knowledge and reasoning regarding the advantages of using this method:

> 2.4 In which ways may students benefit from learning and using the undoing method? Elaborate two major benefits:

The representative answers were also organized into the three basic categories. A new category was created for those participants who claimed "not familiar with" or 'not sure" about this method and hence did not provide an answer.

1. Answers focusing on the mathematical subject matter

It is another method that stresses inverse operations

- It solidifies the concept of inverse operations and order of operations
- It sets stage for studying functions and inverse functions
- It reinforces isolating the variable
- It has simple, basic rules that can be used for different types of equations
- A logical unrevealing of the equation to be solved
- It reinforces reverse order of operations
- It reinforces how to convert standard equations into slope-intercept form
- It is a sequential and logical thought
- Working backward is an important thinking skill
- It helps learning how to solve system of equations
- It connects to elementary school problem solving skills
- It is a concrete relation to arithmetic

2. Answers focusing on student understanding

- It breaks down the equation to a sequence of steps to better understand the process
- It helps visualize where the numbers come from and why you do each operation
- It helps them recognize quicker that when a coefficient and variable are together, they are multiplied together
- They may understand how solving for *x* is the reverse of evaluating an expression
- There is less room for error
- Less likely to mess up with signs when changing sides
- It is easy to visualize
- There is a set of steps to follow
- It is a concrete way to get to the abstract
- Students who cannot do the balancing method might find it useful because you are not working with *x*'s
- Easily checked – plug number in the equation / or using graphing calculators
- Can be replicated easily at home – parents might understand better
- It helps to improve thinking skills

3. Answers focusing on pedagogical representations

- It reinforces the balancing method without using manipulatives

Table 5.16 is the distribution of different types of answers:

Table 5.16 Summary of responses to Question 2.4

Focus	Frequency	Percentage
Learner conceptions	72	50.0
Subject matter	49	34.0
Didactic representation	4	2.8
"Not sure"	19	13.2
Total	144	100.0

25 participants (34.7%) gave two reasons that are both focused on learner conceptions.15 participants (20.8%) had two answers both focusing on the subject matter. The rest participants gave two answers that were of mixed types.

The above two questions and results revealed some patterns in teachers' evaluations of mathematical methods. At the global level, teachers have varied preferences for, or, put different weights on, the three forms of mathematical knowledge for teaching (i.e., learner conception, the subject matter, didactic representations) and some others types of knowledge. Overall, the frequencies and percentages of teachers' responses to the two questions seem to suggest that teachers rely more heavily on their knowledge about students' understanding, than on the mathematical matters. When we zoom into each category of their responses, some subgroups may emerge from the variety of answer provided by the participants. For example, the Leaner Conception category could be divided to include the following more specific categories:

- Making sense of concepts and processes
- Making connections between the abstract and concrete

- Applying a method, carrying out a procedure
- Mistakes and difficulties
- Relating concepts and processes to real life situations
- Thinking skills and problem solving skills

Strategies for improving student understanding

Besides typical student mistakes and difficulties, assessment question 1.2 also asks the participants to correspondingly provide strategies for helping students improve their understanding. Overall, 136 strategies were suggested by the participants for the 138 reported cases of mistakes and difficulties. These strategies were categorized into seven major types by their pedagogical characteristics. The number of cases in each category and its percentage are shown in Table 5.17:

Table 5.17 Summary of responses to Question 1.2 Part 1

Major strategies for helping students	Frequency	Percentage
1. Use visuals aids, hands-on or metaphors to explain	47	34.6
2. Remind rules or review concepts and methods	28	20.6
3. Emphasize the meaning of a concept, symbol, or property	19	14.0
4. Do more practice	13	9.6
5. Rewrite a term or the equation into alternative forms	12	8.8
6. Double-check answers	9	6.6
7. Go through the process and discuss each step	8	5.9
Total	136	100.0

As discussed in the previous section on teachers' knowledge of student conceptions, I categorized all 138 reported cases of student mistakes and difficulties into eight types. For each category of mistake and difficulties, the distribution of various types of strategies varies. For example, the largest category, which involves negative sign, subtraction, and additive inverses, has 47 cases (or 34.1%). Correspondingly, 45 strategies were elicited. Their frequencies and percentages are shown in Table 5.18 below:

Table 5.18 Summary of responses to Question 1.2 Part 2

Major strategies for helping students	Frequency	Percentage
Rewrite a term or the equation into alternative forms	12	26.7
Use visuals aids, hands-on or metaphors to explain	11	24.4
Emphasize the meaning of a concept, symbol, or property	5	11.1
Go through the process and discuss each step	5	11.1
Remind rules or review concepts and methods	5	11.1
Double check answers	4	8.9
Do more practice	3	6.7
Total	45	100.0

Similarly, question 4.3 asked the participants to provide strategies for helping the 128 cases of students with their mistakes and difficulties in using the quadratic formula. A total of 120 strategies were collected. They fell into the same seven categories as for the balancing method (see Table 5.19). The distribution of strategies across the categories is different from that for the balancing method, but the top five categories remain the same:

Table 5.19 Selected responses to Question 4.3

Major strategies for helping students	Frequency	Percentage
1. Emphasize the meaning of a concept, symbol, or property	29	24.2
2. Remind rules or review concepts and methods	26	21.7
3. Use visuals aids, hands-on or metaphors to explain	22	18.3
4. Rewrite a term or the equation into alternative forms	20	16.7
5. Do more practice	12	10.0
6. Go through the process and discuss each step	6	5.0
7. Double check answers	5	4.2
Total	120	100.0

.

Selecting and sequencing mathematics topics

Another topic is the method for solving quadratic equations. As analyzed in the algebra 1 textbook reviews in Chapter 2, completing the square is one of the key methods for solving quadratic equations, and lays the foundation for generating the quadratic formula. Assessment questions 4.1 probes if teachers understand such derivation. As reported earlier, 35 participants (48.6%) correctly pointed out completing the square as the major method underlying the derivation of the quadratic formula (three points), while 33 (45.8%) provided completely false answers or no answer (zero points).

Question 4.2 asks whether and how the teachers would actually teach the derivation process to their first year algebra students:

> 4.2 If you were to teach the quadratic formula, which of the following approaches would you prefer?

☐ **A**. Demonstrate how the formula is derived, expect students to understand and remember each step.

☐ **B**. Demonstrate how the formula is derived, expect students to remember the main ideas only.

☐ **C**. Explain the main ideas behind the formula, expect students to remember and use the formula only.

☐ **D**. Introduce and use the formula directly, explain how the formula is derived in late chapters.

☐ **E**. Introduce and use the formula directly, without ever explaining where the formula comes from.

Please elaborate why you prefer the approach you selected above:

For question 4.2, both the participants who scored three points and those who scored zero points on question 4.1 made diversified choices (see Table 5.20). No outstanding patterns could be directly observed from the data in terms of whether teachers' teaching preferences are relevant to their levels of understanding of the quadratic formula. An overall commonality between the two groups is that the highest percentages of teachers prefer to explain either the entire process or the main ideas underlying the derivation, and expect their students to do no more than remembering the main ideas.

Table 5.20 Summary of responses to Question 4.2 in relation to Question 4.1

Score on 4.1 / Choice for 4.2	3 points		0 points	
	Frequency	Percentage	Frequency	Percentage
A	1	3.1	0	0.0
B	10	31.3	9	29.0
C	13	40.6	9	29.0
D	3	9.4	6	19.4
E	5	15.6	7	22.6
Total	32	100.0	31	100.0

Similarly, no obvious connection was observed between teachers' scores on question 4.1 and their explanations for why they made certain choices in 4.2. When I separate the participants by their choices in 4.2 and compare their explanations across the choice groups, the contrast in their thinking appears: for those 15 teachers who selected A or B and did provide explanations, 13 made statements such as "I want to make connection with the main ideas underlying its derivation so that the students understand where it comes from," "It is good for those who need to know how thing were developed and go on to use it in higher-level courses." For those 15 participants who selected D or E and did provide explanations, nine expressed their concerns such as "The derivation will make students confused," "It's too intense (or too complicated)," "There is not enough time," "The derivation is only a skill, not knowledge," or "Students don't care." These suggest that the first group of teachers focuses more on the longitudinal development and connections among mathematics concepts and processes, while the second group of teachers thinks more about the barriers in student prior knowledge and readiness.

Strengths and limitations of using manipulatives

In the follow-up interviews, the eight participants were asked about the strengths and limitations of using balance scales and algebra tiles in teaching and learning equation solving. Their characterizations covered learner characteristics, the representational and mathematical features of the manipulatives, other chatatersitics, and some other considerations. Below is a summary:

1. Learner characteristics

Strengths: the participants mentioned that there are different types of learners, for those kinesics learners or lower level kids, the manipulatives help them to "see the pieces" and "see what's going on." They "instill pictures in their heads." "Students enjoy using their hands a lot." When they play something in hand, "it keeps them paying attention," and helps them to "retain information."

Limitations: the students get bored after playing with the manipulatives for a while, many honor students do not want to deal with it, they "find it confusing," they "hated it," "they want to do some more complex thing with them."

2. Representational features

Strengths: The manipulatives are visual and concrete, it's "another way of representing information." It is similar to the Cuisenaire rods that many students have used before. It's versatile, can be used for following concepts.

Limitations: Students may be overloaded by extra information in using the manipulatives, because students need to make connection between physical objects (color, shape) and their algebraic meanings (1, x, positive, negative, etc.). And they need to learn how to use. It may become "a whole thing the students have to learn." Some students "got caught up in the mechanic of using" the manipulatives.

3. Mathematical features

Limitations: the manipulatives can represent only integers, not decimals. They ca not represent cubic or quadric equations. When negative numbers are involved, it can get "clumsy." Especially for subtracting negative numbers, it is an artificial and tedious process. If the students do not have a firm grasp of operation of integers, the manipulatives will "give them hard times."

4. Other characteristics

The manipulatives are used in school only, not in daily life, students ca not take them back home. A school may not be able to put enough manipulative for all students in each classroom, sometimes the students are fighting over for it.

5. Other considerations

Teachers' uses of manipulatives are also related to their teaching goals and beliefs: "the ultimate goal is to go beyond physical manipulation and understand the real math," "the whole point is to be able to do the abstraction and higher level thinking," "at some point, they're gonna have to realize that's a tool for learning but my learning have to go further than just being kinesics and being able to touch and feel," "using manipulatives helps students know why's, but no why's or too much why's are not good."

As a result of the above understanding and considerations, most participants do not want their students to rely on manipulatives in solving equations. The manipulatives are mainly used for "demonstration," "warm up," or tutoring "lower level kids" who do not understand the symbolic processes well.

Typically, teachers began their characterizations with the representational features or student learning characteristics. They tend to mention about the mathematical features (mostly pitfalls) at the end, especially through my reminders such as: "is there any kind of equations that can't be solved with this manipulative?"

Teachers' Conceptions of the Features and Roles of Multiple Strategies

In the follow-up interviews, the participants were asked a sequence of questions regarding five algorithms and strategies for solving linear equations: the balancing method, the undoing method, tracing the line, examining the table, and intersecting the lines.

Goals of teaching multiple algorithms and strategies

At the beginning of the interviews, I briefly discussed with the participants the different methods they have taught for solving linear equations, then asked them what their expectations are in teaching multiple methods, for example, do they hope all students can grasp a most important method, or every student can choose whatever method he or she likes?

The responses from four of the interviewees demonstrated three major types of thinking.

1. Focusing on the formal method

Both Amy and Pam prefer to focus on the balancing method out of practical concerns, but the specific reasons they gave are quite different. Amy believed that, "from a purely utilitarian aspect," students need to be able to use the balancing method to solve symbolic equations which are modeling real world problems. "First and foremost, students need to understand the balancing method which is extremely important…other methods are enrichment."

Pam teaches students in the magnet program of her middle school. After graduation, most of her students will go to a nearby high school which is one of the top-ranked in the school district, so she felt that "I need to make sure they can do the symbolic way. I hope they can do both, but I know they really need to master the symbolic manipulation, because of where they are going."

2. Exposing students to multiple strategies and let them choose

Rene and Tom shared two common beliefs. On the one hand, students learn differently and have different preferences. On the other hand, many of them only want to stick to a single routine, they get frustrated or confused when learning multiple methods. Because of these, Rene believed that students will benefit from multiple strategies:

> The advantage is… in terms of broad conceptual knowledge of the students. I think they need to see that you can do the same thing in different ways. There's something they need to accept and realize and become mature about it. Part of the maturation process, growing up in high school, is to learn different aspects of things. I can teach them one way, they memorize and follow through, but do they really understand what it means? If they see different ways, it broadens their conceptual knowledge.

Her expectations of learning multiple strategies differ across students:

> My bottom line is that they learn how to solve an equation. As long as they learn one method I'm happy. But I guess it's a matter of differentiating your classroom. There's got to be some students in there who will benefit from seeing another method and expand their knowledge. In general, it will really help them in the future, have a broad conceptual idea. For the kind of students that are gonna to go to calculus, if they are not seeing different methods in 9th grade they are not able to go to calculus.

Tom's expectation is a bit higher than Rene's, in that he hopes the students are familiar with all methods before selecting their own preferences.

> The student has to be familiar with all of the ways of expressing this, but as to which one they prefer, they are each gonna prefer their own way. Some lean themselves more towards a graph, some lean more towards a table, so the students have to get very familiar with one method and use it almost exclusively.

3. Selecting an optimal method for individual problems

Following his expectation of students pick their own preferred method, Tom specified a higher level goal – students get to know which method works better for a certain situation:

> They are going to learn… this problem really works better with a table than does graphing, or this problem works better with analytically than a table because… for example, it may not have an integral solution, so the table may not be the best way to approach… so they learn to recognize those types of details.

In summary, the participants considered both learner characteristics and the structure and inter-connection of the mathematics system. Their expectations ranged from concentrating on one basic method to developing students' skills and flexibilities in selecting optimal methods for various problem situations. These three types of expectations are not necessarily hierarchical, but overall their levels of complexity do seem to be increasing.

Similarities among various methods

The participants were first asked to find which of the five methods are more similar to each other than to the rest, from four different perspectives:

1. By mathematical nature

Two participants, Mary and Teresa, thought that tracing the line and intersecting the lines methods are most similar since they are both about finding a point, or an ordered pair. Here they might have overlooked the subtle difference between the two methods: one is about a point, the other is only about the *x*-coordinate of the point.

Yvonne initially suggested examining the table and intersecting the lines, because "you are gonna find the same value of *x* and *y*." Later she changed to examining the table and tracing the line, "…because that's when you change the *x*, what happens to the *y*… you are moving towards a point, and you are gonna see it in the table… its about input to output…" This choice is consistent with the one she picked later from didactic representation perspective.

Pam also claimed examining the table and tracing the line: "…on a table you are just looking for the point of intersection on the two graphs, they are both looking for the same x values for the same y values…"

Both Jane and Rene found two groups by similarity: (a) examining the table and intersecting the lines and (b) the balancing method and undoing method. Jane simply thought that the first pair is more "visual" and the second pair is "more process-oriented." Her explanation was actually based on her student learning. Rene also relied on her students' thinking for (a): "I guess I'm just programmed like my students, the table is just another representation of the intersection." And for (b), she was not able to make distinctions between the balancing and the undoing method even after I explained to her.

In summary, the teachers drew a line between two groups of method: (a) the balancing and the undoing methods which are symbolic processes and (b) tracing, using table, and finding intersection point, which are visual and mostly associated with graphing tools. The similarities between methods could either come from the processes (changing the *x* and changing the *y*, or moving towards a point), or from the end results (the intersection point, an ordered pair, or "the same *x* and the same *y*"). Further, teachers' mathematical comparisons may actually reflect their experiences with learner conceptions or pedagogical approaches.

2. By the meaning of equality

The participants were asked whether the meanings of and equality or the equal sign are all the same when we apply the five methods.

Teresa, Tom, and Amy believed the meanings are all the same. While Teresa simply stated "equal always means they are balanced so the meanings are the same," Tom used "sameness" to give a broad characterization of all cases:

> I think the meaning of the equal sign is "this is the same as this." So I treat it as sameness, rather than a true equality. Because if I'm looking at the lines, where is this line the same as this line? They are the same at this point. I'm not necessarily thinking the lines are equal, but they are the same, i.e., that's where they intersect.

Amy interpreted the sameness from a different perspective:

> I think they are all just the same. They are probably what we refer to as multiple representations. It's basically a slightly different way of dealing a particular problem, but its really of course the same problem and the same method. They just look slightly different, we are still, in all of them, using the idea of balancing and opposite operations. So I would say the role of the equal sign doesn't change in any of the problems.

Mary and Rene claimed the balancing method and intersecting the lines assign the same meaning to equalities because they both indicate two equal values. Jane thought they were using table and intersecting the lines, with a similar reasoning.

Same as looking from the mathematical perspective, Yvonne still thought tracing the line and examining table involve the same kind of equality: linking from input to output.

In this case, the participants either believed all of the solving methods guarantee the same connotation for all equalities, or used similar reasoning in finding similar pairs as they did from the mathematical perspective. Two participants did mention students' immature notion of the equal sign (as command of an operation), and the view of equality as equivalence between quantities or expressions. These will be discussed below under student conceptions.

3. By student conceptions

In finding similar methods from a mathematical perspective, Rene already mentioned that her students were "programmed" to "interchangeably" use the table, tracing, and intersecting lines methods. From student learning perspective, she still thought the three methods can be "put together," because "it's the calculator idea."

Mary's answer is based on the teaching and learning in her classroom:

> I would say filling the table and intersecting the line method. Because they are very closely related, and we do use the TANG, so that they can take it from a table and draw one equation, and they can also draw anther equation and then find the intersections as well.

Here TANG refers to Table, Analytical word problem, Numerical equation, and Graph, which are four aspects of studying linear functions or linear equations in two variables, and are part of the algebra curriculum being implemented in Mary' school district.

Pam began with analyzing the undoing method and a popular notion used by her students:

> … the reason why I don't really like the undoing is because the kids will think equal where the things are unequal, like say $5 - 2 = 3 \div 3 = 1$, it's not equal. When you're undoing, the equal has no meaning at all, that's what you are undoing stuff, which is on the other side, is what you are actually performing your undoing on… and the way the kids write it is just horrible. Use the properties of equality to have mathematically equal to each other, they are equivalent and then just have to treat that equal, and then, the Word of Law, you know, whatever you do, you have to maintain the equality.

Further, Pam clarified the nature of undoing or the students' wrong notions:

> It's like simplification. So you have the mathematical expression, like 3+2 and you simplify it to get 5, yeah… that's a different thing of the equation, 'cause with an equation you are given two things, and you need to find what the unknown is. Totally different things. Simplification, solving are different…
>
> One of the biggest hurdle for me, it's getting them to understand the difference between simplifying and solving, and so when they distribute they sometimes think they have to do it to both sides, and I'm like "No… Distribution is not… it's just… a simplification tool that gives you something equal so that it doesn't change the value of the variable, so there's no need doing to both sides."

Eventually, Pam made a conclusion about student thinking:

> My kids would think that balancing and undoing are the same. 'Cause they resist so much… from the undoing to the balancing. So, to me, I think lots of kids think that they are the same, which is not good. 'Cause when they get to the complex equations, they can't do the undoing.

Jane also sensed the special meaning of the undoing method:

> The undo…even though they use them together with the balancing, I don't think the feel for the equal sign… I'm not saying its wrong, but it's different…
>
> …Because you are not really keeping stuff equal… you are undoing a process. Are the kids really sensing the equality? I don't think so. I think you could…but if

> you do, I think you are really doing something like the balancing…the actual undo is more like order of operation, an undo process…

She continued to comment on students' misconceptions about equalities:

> "They are the same"…that's part of the problem…the whole idea of being the same, sometime get in their way…sometimes they don't get that it's the value that are the same…you know, you can have $2a = 3b$, that doesn't look the same…but it means whatever you have on the left, that has the same value as what's on the right… So I'm not sure our kids really have… not sure if they have a good concept of equality, just like they have a hard time with equivalence… 5/10 is the same as a half, why? Because I can divide the top and the bottom by 5… Well…no.. they are the same because they represent the same part of a whole. They struggle with that in middle school.

4. By ways of teaching

Yvonne, Pam and Mary all picked table and intersecting lines as most similar from teaching perspective. Yvonne's judgment come from her actual teaching experience: "What I have done is to combine the two using split screen with graphing calculators, so they see what's happening to the values and the table." Mary was also consistent with her previous description of student learning experiences (the TANG). Pam's reason was simply that the two methods are two ways of looking at the same process.

In making her choices, Jane referred back to the Process-Visual grouping that she defined previously from mathematical perspective. Rene insisted again the balancing and undoing methods since she still could not tell the difference between the two.

Overall, the participants selected similar methods from various angles. A major distinction exists between the symbolic methods and the graphical ones. Some participants' choices are more connected and coherent than those of some other participants. The participants did not show strong knowledge about the subtle meanings of equality underlying the different equation solving processes.

Features of mathematical algorithms and strategies

The participants were provided with eight measures and asked to rate the five methods for solving linear equations on a five-point scale, with 1 representing the lowest extent, and 5 representing the highest. For method and each measure, Table 5.21 shows the eight participants' average rating in the corresponding cell. For each measure, the mean value and mean difference of the five ratings are then calculated and entered in the "Average rating" and "Mean difference" columns, respectively.

Table 5.21 Summary of ratings on the five methods

	The balancing method	The Undoing method	Tracing the line	Examining the table	Intersecting the lines	Average rating	Mean difference
1. Accuracy	**4.8**	4.1	2.6	2.6	3.1	3.5	0.8
2. Generality	**4.5**	2.4	3.0	3.9	4.0	3.6	0.7
3. Efficiency	**4.3**	3.5	3.5	3.0	3.6	3.6	0.3
4. Transparency	3.5	**3.8**	3.4	**3.8**	3.4	3.6	0.2
5. Mathematical value	**4.5**	3.8	3.8	4.4	4.4	4.2	0.3
6. Easy to apply	**4.4**	4.0	3.6	3.3	3.1	3.7	0.4
7. Easy to teach	**4.1**	**4.1**	4.0	3.9	3.9	4.0	0.1
8. Easy to learn	3.6	3.6	**4.3**	3.9	3.9	3.9	0.2

The bold numbers indicate the maximal average rating for each measure.

Although each participant rated quite differently, the average ratings show that the teachers as a group had quite appropriate evaluations of the methods by features:

The balancing method was rated the highest on six measures. It can represent the answers (real numbers) in the most accurate way. Tracing the line, examining the table, and intersecting the lines are less accurate because they can only represent real numbers with up to nine decimal digits.

In terms of generality, the balancing method can be applied to any linear equation, while the undoing method and tracing the line method can be only directly used to solve for $ax + b = c$ type of equations.

As shown by the mean differences, rating the methods by accuracy and generality yields the highest level of variances.

The most efficient method is balancing. It only involves step-by-step symbolic transformations and numeric computations. Examining table, as several participants pointed out, may take quite some time to adjust the increment or to move up and down the rows before the solution could be located or approximated.

The undoing method and examining the tables have the highest transparency. The undoing method is very close to students' typical notion of equations (commands and results) developed in elementary school, and examining the tables involves mostly numbers.

For mathematical value, the symbolic, tabular, and graphical approaches got higher ratings. They are the most basic and interconnected approaches to the study of equations, and have been promoted by mathematics education reform, particularly, the Texas state mathematics curriculum standards.

The balancing method is the easiest to apply, while intersecting the lines is the hardest. This is due to the simple fact that one relies on graphing tools while the other one doesn't.

The average ratings on "easy to teach" are the most homogeneous group. By checking the original ratings of each participant, I realized that there is the divide between two groups of methods: (a) the symbolic group: the balancing and undoing methods and (b) the "visual" group: tracing, table, and intersecting the lines. Some participants rated the first group higher than they did on the second group, some other participants did the opposite. This may explain why the five averages become quite close.

Tracing the line method became the easiest to learn. It is likely that the participants feel it is not as complicated as the rest four methods in symbolic manipulation, numerical exploration, or graphical manipulations and interpretations.

At the end, the participants were asked to select three of the measure which they believe are most crucial. The measures and the total number of times that each measure was selected by a participant are as follows: Accuracy (6), Mathematical value (5), Easy to learn (4), Transparency (3), Generality (2), Efficiency (2), Easy to apply (1) and Easy to teach (1).

Teachers' Conceptions of the Roles of Mathematical Routines

The participants were asked about the pros and cons teaching and learning mathematical routines, procedures, and rules, and why they would teach multiple strategies for carrying out procedures.

Roles of mathematical routines and procedures

To a large extent the eight participants shared common knowledge and beliefs about the pros and cons of teaching and learning mathematical routines and procedures. Nonetheless, I organize them into four categories based on the subtle differences in how the participants think procedural knowledge and conceptual understanding should be connected or reconciled.

1. Learners at different levels understand routines in different depths

While believing that mathematical routines and procedures are good to all students as the starting point, and having tried to teach routines for conceptual understanding, Tom distinguished between his upper level students and regular (or younger) students:

> For my upper level students it's a good starting point, then we go, why are we do this first step as we always do? What happens if we change the order of the steps? What problems do we run into? So that's what I really headed to with my honor class with my algebra 1 students. With my regular students I'm happy if they solve regular equations, they are not interested in why are we doing this first… so we will do an equation like $3x + 4 = 5$ what happens if we divide by 3 first? Oh, we get these fractions, I will never do it again. So they get the idea of why we do certain thing… but the hierarchy… I think the younger people need a routine. It makes them feel better… OK... step 1, step 2, step3…

Further, he admitted that following routines may be the best he could expect for his low level students:

> I want my students to understand the mathematics behind what we are doing, but… some of our students aren't going to be there, especially my lower level students. They are not gonna understand why we do multiplication and division at the same time. So they are not gonna understand why you have to have your constant terms taken care of first before worrying about your linear terms. And so, for those students you need those hierarchy, you do this first, you do this next, then you'll have a solution.

2. Procedures first, understanding and discovery later

Based on her experiences as learner herself and with her students, Amy believed that conceptual understanding and flexibilities are the ultimate goals, but sometime they have to come after procedure fluency:

> Sometimes we do have to tell a student this is how it is, and I need you to practice this, and the true understanding will come later. So one of the cons of something very procedurally based is that it sacrifices, initially, understanding, just so that we can understand the procedure. So the understanding sometimes has to wait, till we get the procedure done. On the other hand, one of the pros is, if the students know the procedure, then what can follow is an understanding of the procedure. And after that, and understanding of the underlying math of the procedure, and they will be in a better position to apply that to something slightly different…And then after that we can do investigative…we can change some of the variables, so to speak, what happens if we change this, change that… they can start to do some discovery on their own.
>
> One of the things I tell them, memory doesn't mean you have learned anything… but it's a gateway to learning, pathway… When I was in school, sometimes I memorized without understanding, but with examples, the memorized began to make sense. Sometimes the understanding of what we are doing right now doesn't come until several weeks, several months or even several years down the road. But a lot of it begins with a procedure. If I have to put a tally one pros and cons I would say we have many more pros than cons.

Amy's emphasis on procedural knowledge also comes from her understanding of the nature of mathematics:

> Math is about finding patterns, predict…describe… if you are good at finding patterns you are good at math…some students are good natural pattern finders, some are not, I think the procedure helps…We do have to have room there for discovery. But procedure is very helpful, it help the students put things in perspective…

In her brief response Mary suggested similar ideas as Amy's: "The routines are organizational, set an anchor, they can always come back… as they can go life part understand and rebuild. It's progressional."

3. Conceptual understanding is crucial

Partially because she is teaching honor students, Pam pointed out the limitations of solely knowing routines and directly emphasized conceptual understanding and problem solving in teaching and learning procedures:

> I think routines are pretty easy, in terms of students being able to solve the exact same kind of problem… I mean, if you want them to practice, you want them to solve quadratics and you show them the quadratic formula… it always gives you the exact answer… they can solve any quadratic, but then it's not teaching them how to think, 'cause just some problems don't have a context, it's not problem-solving, and so… I like deriving the formula with them, I like the kids derive the formula, because that has more meaning and it makes them think. I want them to understand where the answers came from, just having a routine, you don't necessarily understand where the answers come from, what they mean, anything.

4. Finding balance between procedural fluency and conceptual understanding

The other four teachers parallelly compared the pro and cons of procedures, without necessarily defining a clear path for linking procedural and conceptual understanding. A representative account came from by Jane who first examined procedures from the logical and strategic nature of mathematics:

> I love for my kids to think logic… There are so much strategy involved in life, not just in your job. If you learn methods and more strategies to attack problems, it makes you a better person, not just a better mathematician. So I'm a real believer in the whole logic of mathematics and how beneficial it is.

Then she turned to the learning and using mathematics in addressing the downside of following procedures, and concluded with the willingness to look for balance:

> Well if you do too much…you can squash creativity, some people think there is nothing in math about creativity, when there definitely is…There are different ways to do things, there are different approaches, there are totally ways of looking at some things, and so I think we have to balance or all the things can make a child feel like there is no place to be creative in a math class, and that simply is not the case. That is to me the con. That's why the balance is important. I have had a lot of struggle with that balance…

Rene pointed out some pros that are very similar to those mentioned by Jane:

> The pros of teaching routines and procedures: they are logical, we are teaching efficient logic. And I have told my students repeatedly… they say "I never gonna solve an equation like this" …. I said "No, you may not have to solve an equation, but it teaches you a logical way of looking at things that will help with many aspects of your life. So it doesn't matter if you are flipping burgers at MacDonald's or if you are an electrical engineer, thinking efficiently and logically will help you in life."
>
> Routines give students a structure, its comforting to them. We have a lot of students who are unnerved…school is the only structure for them, that consistency…

Yvonne mainly commented on the importance of procedure skills by expressing her disappointment with the fact that many school come to high school without sufficient basic skills in elementary and middle school arithmetic and number sense. She emphasized basic skills with an analogy: "Just like you are driving a car, if you don't have any gas, you are not going anywhere."

In summary, the participants examined the role of teaching and learning mathematical routines and procedures mostly in terms of the nature and application of mathematics. The strengths of routines include: they are logical, they set a foundation for future studies and conceptual understanding, and they provide order, structure and consistency for mathematics study and strategies for real life problem solving. When students learn or follow the routines without enough understanding, it puts constraints on their flexibility in problem solving, limits the development of their high level thinking skills, and deprives their opportunities to explore and make discoveries. The participants had subtly different approaches to blending procedural skills and conceptual understanding.

Flexibility in following routines and rules

In their responses to some of the written assessment questions, some teachers mentioned a few rules in arithmetic and algebra, such as the order of operations (some teachers use the acronym PEMDAS – Parenthesis, Exponents, Multiplication, Division, Addition, and Subtraction), the Golden Rule of Algebra ("What you do to one side of an equation, you should do it to the other side"). A few teachers also emphasized that in solve linear equations, one should always undo addition and subtraction before undoing multiplication and division. I brought up these rules to the interview participants and asked them whether we should always follow such rules in computing and problem solving.

In one way or another, all the eight participants expressed their belief that mathematical rules are not absolute. For example, "There is always flexibility… that's the whole nature of mathematics, that you can follow different act but still reach the same destination" (Tom), "It depends on the problem and students" (Amy), "As long as the students don't violate the property of equality, they can do whatever they want." Five of the participants used PEMDAS as a specific example to support their claims. Tom and Teresa both claimed that M(ultiplication) and D(ivision) should be grouped together since they are inverses, so are A(ddition) and S(ubtraction). Amy, Mary, Jane showed that we do not have to strictly follow the order in cases like computing $3 + 4 + 5^2$ and simplifying $5(x + 2) – 3(x + 2)$. Pam, Rene and Amy gave examples of equations for which we could first divide all the coefficients by a common factor if it is not trivial, or multiply first when one or more of the coefficients are fractions.

Although the participants all acknowledge the flexibilities in using the rules, they also believed in the positive roles of the rules as they did in their discussions about routines and procedures. For some, the reality of teaching and learning even makes the rules a must. For example, Rene said, "in terms of teaching you have to follow the rules pretty much the time… and make it clear to begin with." Tom was still concerned about his lower level students' unwillingness to go beyond the rules:

> …the whole idea here is that the students want the magic wand, they can touch it with the magic wand, steps 1, step2, step 3….If it works for 99% of the problem, they are not gonna see the 1% there's going to be a problem. Ultimately I want them to be beyond that, but…

One issue that I did not pay full attention until after the interviews is how teachers think about the questions "why do we have to multiply and divide before adding and subtracting?" or in general, "why do we have to follow the order of operation?" When I asked this question in the interviews, Teresa answered that "I don't know. If you don't follow you get the answer wrong. That's bad. I don't even know why I would do that." In contrast, Jane went to the opposite end by claiming that the order of operation comes out of daily life and hence should not be called a rule:

> PEMDAS is introduced really early, in elementary schools, so my kids have heard it. But we try to undo that a lit bit, the idea of multiplying and dividing then adding and subtracting… if you work in a restaurant you get paid this much per hour then you get a tip, well, what's the order? You multiply the number of hours by the hourly pay, then you add the tip. It's not a rule, it's just the way real life works. So our district has pulled back a bit from that lately, away from that stupid rules, and change it to... how does that behave in real life? What do we do in real life? And we are finding the kids understand the whole order of operations process much better, because that's just the way it works in life…I think that's a mistake to call everything a rule. Everything that is a rule, order of operation is not one of them. It is the natural flow of how numbers work.

It would be interesting to find out, through future studies, whether teachers understand the nature of the order of operations as a convention, and why real-world situations are not able to justify such a rule.

RELATIONSHIP BETWEEN TEACHER KNOWLEDGE AND TEACHER CHARACTERISTICS

Hypothesis tests were conducted to compare teacher performance by four basic teacher characteristics: college major, course-taking in college mathematics and mathematics education, school algebra course-teaching, and number of years of algebra teaching. Teachers' own accounts were also analyzed in term of the sources of impact on their knowledge and conceptions.

Differences in the Mean Scores between Teacher Groups

Comparison by college major

In this study, there are a total of 36 participants who majored in mathematics, and two participants were non-mathematics majors but with a master's degree in mathematics. Their mean score in the assessment was calculated and compared with that of the rest participants, i.e., those who had neither undergraduate nor graduate major in mathematics. The group statistics is shown in Table 5.22:

Table 5.22 Comparison by college major

Major	N	Mean	Standard Deviation	Standard Error Mean
Mathematics	38	59.1842	14.66353	2.37874
Non-mathematics	34	49.5000	17.11857	2.93581
Total	72	54.6111	16.49119	1.94351

The two mean scores show that, in this study, teachers with a degree in mathematics did score higher than the other teachers. An independent-sample *t*-test was conducted on the following hypothesis regarding the entire population:

H_0: There is no significant difference in teachers' performance between those that were mathematics major and those that were non-mathematics majors.

H_1: There is a significant difference in teachers' performance between those that were mathematics major and those that were non-mathematics majors..

The result is shown in Table 5.23 (the two groups were assumed to have equal variances).

Table 5.23 Results of significance test

T	df	Significance (2-tailed)	Mean Difference	Standard Error Difference	95% Confidence Interval of the Difference	
					Lower	Upper
2.585	70	0.012	9.68421	3.74598	2.21309	17.15533

At $\alpha = 0.05$ level, we reject the null hypothesis H_0, i.e., teachers with a college degree in mathematics scored significantly higher on the assessment than those with other majors.

Comparison by advanced mathematics course-taking

Besides college major, specific course-taking is another measure of teachers' mathematics background. It is reasonable to assume that, in general, mathematic majors take more college mathematics course than other majors do. To verify this assumption for this study, I ran a correlation analysis between the participants' college major and advanced mathematics course-taking. The result is shown in Table 5.24. The * indicates that the correlation is significant at α = 0.01 level (2-tailed).

Table 5.24 Correlation between major and total mathematics courses

		Total Math Course	Major
Total Math Courses	Pearson Correlation	1	0.569(*)
	Sig. (2-tailed)		.000
	N	72	72
Major	Pearson Correlation	0.569(*)	1
	Sig. (2-tailed)	.000	
	N	72	72

We have found that college major may predict teachers' performance, and college major and advanced mathematics course-taking are confounded measures in teachers' mathematics background. So naturally, I wonder if advanced mathematics course-taking also makes significant difference in teachers' knowledge.

Descriptive statistics shows that the participants' advanced mathematics course-taking (as listed in the questionnaire) ranges from 0 to 7, with a median of 4 and an average of 4.5. Table 5.25 shows the distribution of the number of mathematics courses:

Table 5.25 Summary of advanced mathematics course-taking

Number of Courses Taken	Frequency	Percent	Cumulative Percent
0	3	4.2	4.2
1	4	5.6	9.7
2	7	9.7	19.4
3	5	6.9	26.4
4	12	16.7	43.1
5	17	23.6	66.7

6	13	18.1	84.7
7	11	15.3	100.0
Total	72	100.0	

A linear regression analysis yields the scatter plot and the regression line (see Figure 5.20) and the model summary (see Table 5.26).

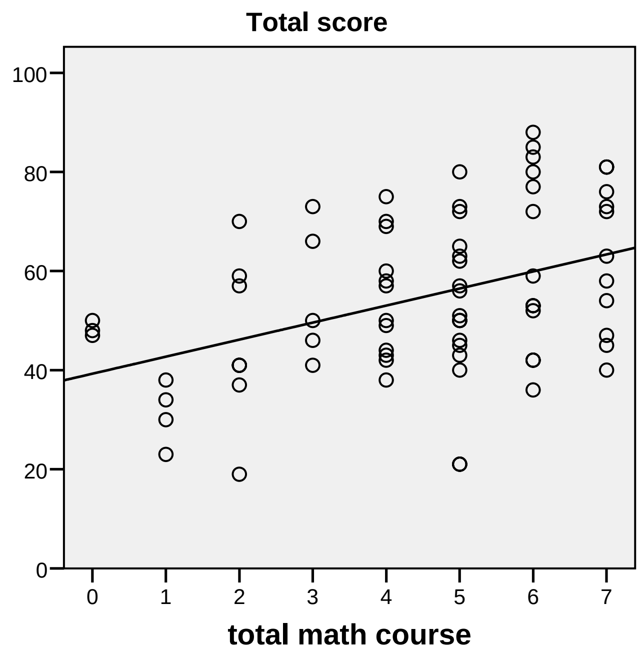


Figure 5.20 Scatter plot and regression line

Table 5.26 Model summary of regression

R	R Square	Adjusted R Square	Std. Error of the Estimate
0.406	0.165	0.153	15.17480

Tables 5.27 and 5.28 show that, when we use the total number of advanced mathematics course as a predictor for a teacher's total score in the assessment, the sum of squares and the regression coefficients are significant at 0.01 level, i.e., when the number of courses is no more than seven, the number of mathematics courses is a significant predictor of teachers' performance (each additional course would contribute to about 3.4 points in the score).

Table 5.27 Analysis of variance

	Sum of Squares	df	Mean Square	F	Sig.
Regression	3189.902	1	3189.902	13.853	0.000
Residual	16119.210	70	230.274		
Total	19309.111	71			

Table 5.28 Regression coefficients

	Unstandardized Coefficients		Standardized Coefficients		
	B	Std. Error	Beta	t	Sig.
(Constant)	39.283	4.490		8.749	0.000
Total math course	3.438	0.924	0.406	3.722	0.000

Comparison by mathematics education course-taking

Teachers' performance was compared between those who have taken no more than one mathematics education course and those who have taken more than one course (see Table 5.29):

Table 5.29 Descriptives of performance by mathematics education course-taking

Math Education Course-taking	N	Mean	Standard Deviation	Standard Error Mean
0 – 1	38	52.6053	15.70426	2.54757
2 – 3	34	56.8529	17.28571	2.96448
Total	72	54.6111	16.49119	1.94351

The two mean scores show that teachers who have taken two or three mathematics education courses scores higher than those who took fewer courses. This may be related to the fact that teachers who have taken more mathematics courses also tend to take more mathematics education courses (see Table 5.30):

Table 5.30 Distribution of teachers by mathematics and education course-taking

		Total math education courses		Total
		0 – 1	2 – 3	
Total math courses	< 5	17	11	28
	>= 5	21	23	44
Total		38	34	72

An independent-sample *t*-test was conducted on the following hypothesis regarding the entire population:

H_0: There is no significant difference in teachers' performance between those who have taken two or more mathematics education courses and those who have taken fewer courses.

H_1: There is a significant difference in teachers' performance between those who have taken two or more mathematics education courses and those who have taken fewer courses.

The result is shown in Table 5.31 (equal variances were assumed):

Table 5.31 Results of significance test

T	df	Significance (2-tailed)	Mean Difference	Standard Error Difference	95% Confidence Interval of the Difference	
					Lower	Upper
-1.093	70	0.278	-4.24768	388772	-12.00150	3.50614

At $\alpha = 0.05$ level, we fail to reject the null hypothesis H_0, i.e., there is no significant differences in teachers' performance between those who have taken two or more mathematics education courses and those who have taken less courses.

Comparison by school algebra course-teaching

Teachers' experiences with teaching school algebra courses may have multi-dimensional influences on teachers' knowledge for teaching algebra and its topics. Based on the participants' responses to the survey item on the algebra courses they have taught in the last five years, I divide them into three groups: 1. Pre-algebra: those who have taught first year algebra before but are mostly experienced with teaching pre-algebra; 2. Algebra 1: those who are mostly experienced with first-year algebra and have not taught any algebra course beyond that (second-year algebra, advanced algebra, etc.); 3. Advanced algebra: those who have taught or are mostly experienced with courses beyond first year algebra.

Group statistics shows that the mean scores of teachers in Group 2 and Group 3 are higher than that of the teachers in Group 1 (see Table 5.32):

Table 5.32 Descriptives of performance by algebra course-teaching

Group	Algebra Course-teaching	N	Mean	Standard Deviation	Standard Error Mean
1	Pre-algebra	12	39.8333	15.16475	4.37769
2	Algebra 1	34	59.7941	14.77668	2.53418
3	Advanced	26	54.6538	15.54334	3.04830
	Total	72	54.6111	16.49119	1.94351

A one-way ANOVA test was carried out to examine if the differences in the mean scores among the three groups are significant:

H_0: There is no significant difference in the mean scores among the three groups.

H_1: The mean scores of the three groups are significantly different.

The result is shown in Table 5.33:

Table 5.33 Results of one-way ANOVA test

	Sum of Squares	df	Mean Square	F	Sig.
Between groups	3534.001	2	1767.001	7.729	.001
Within groups	15775.110	69	228.625		
Total	19309.111	71			

At $\alpha = 0.01$ level, we reject the null hypothesis H_0, i.e., there is significant difference in the performance among the three groups of teachers who have taught different types of algebra course.

Post hoc multiple comparisons were then conducted on the mean scores of the three groups with the Bonferroni method. This method controls the family-wise Type-I error (i.e., mistakenly rejecting a true null hypothesis) of multiple comparisons by setting the confidence level for each comparison at $\alpha^* = \alpha/N$, where α is the overall confidence level and N is the total number of comparisons (Dean and Voss, 1999; Lomax, 2001). The results (see Table 5.34) suggest that, at $\alpha = 0.05$ level, the mean scores of the Algebra 1 group and the Advanced group are both significantly higher than the mean score of the Pre-algebra group (the numbers with * in Table 5.34):

Table 5.34 Post hoc comparisons by course-teaching

Groups (I)	Groups (J)	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Upper Bound	95% Confidence Interval Lower Bound
Pre-algebra	Algebra 1	-19.96078(*)	5.07704	.001	-32.4185	-7.5030
Pre-algebra	Advanced	-14.82051(*)	5.27687	.019	-27.7686	-1.8724
Algebra 1	Advanced	5.14027	3.93923	.589	-4.5256	14.8061

Comparison by years of algebra teaching

As mentioned in the research design, the participants' total number of years of algebra teaching ranges from 1 to 30 years, with an average of 8.3 years and median of 7 years. Table 5.35 shows a more detailed descriptive statistics:

Table 5.35 Summary of years of algebra teaching

Year of algebra teaching	Frequency	Percent	Cumulative Percent
1	4	5.6	5.6
2	2	2.8	8.3
3	6	8.3	16.7
4	4	5.6	22.2
5	7	9.7	31.9
6	9	12.5	44.4
7	6	8.3	52.8
8	6	8.3	61.1
9	5	6.9	68.1
10	5	6.9	75.0
11	6	8.3	83.3
12	1	1.4	84.7
13	2	2.8	87.5
15	2	2.8	90.3
16	1	1.4	91.7
17	2	2.8	94.4
20	1	1.4	95.8
24	1	1.4	97.2
25	1	1.4	98.6
30	1	1.4	100.0
Total	72	100.0	

By using five and nine as the cut points, the participants were divided into three groups: 1. Those who have taught algebra for no more than five years; 2. Those who have taught for six to nine years; 3. Those who have taught for 10 years or longer. The group statistics is shown in Table 5.36:

Table 5.36 Descriptives of performance by years of algebra teaching

Group	Year of Algebra Teaching	N	Mean	Standard Deviation	Standard Error Mean
1	<= 5	23	50.9130	14.46285	3.01571
2	6 – 9	26	59.4615	15.17691	2.97644
3	>= 10	23	52.8261	19.00874	3.96360

Teachers in Group 2 performed better than those in Group 1 and Group 3. In other words, teachers who have taught algebra for six years or longer performed better than those that have taught for less than six years. A one-way ANOVA test was carried out to examine if the differences in the mean scores among the three groups are significant:

H_0: There is no significant difference in the mean scores among the three groups

H_1: The mean scores of the three groups are significantly different.

The result is shown in Table 5.37.

Table 5.37 Results of one-way ANOVA test

	Sum of Squares	df	Mean Square	F	Sig.
Between groups	999.519	2	499.760	1.883	.160
Within groups	18309.592	69	265.356		
Total	19309.111	71			

At $\alpha = 0.05$ level, we accept the null hypothesis H_0, i.e., there is no significant differences in teachers' performance among those who have taught algebra for various amount of time.

Teachers' Own Accounts of Relationship between Knowledge and Experiences

Influences of professional development activities

At the end of the academic background questionnaire, the participants were asked to rate the extent to which professional development activities have been helpful to them in various aspects of algebra teaching and learning. The scale is three-point: 0 – Not at all, 1 – Somewhat, and 2 – Very much. The nine aspects are list below (see Table 5.38) by the order of the 72 participants' average ratings:

Table 5.38 Summary of responses to survey Question 8

Aspects of learning	Average rating
Using manipulative or technologies in teaching algebra	1.46
Preparing students for district and state assessments	1.31
Methods for teaching algebra	1.14
NCTM and state standards specific to algebra	1.04
Methods for assessing student learning in algebra	0.90
How students learn algebra	0.90
Teaching algebra to students with diverse backgrounds	0.88
Algebra-related advanced mathematical knowledge	0.87
Algebra textbooks and other curricular materials	0.85

Each participant's ratings on all of the aspects were then summed up, and the correlation statistics were computed (see Table 5.39). Interestingly, at $\alpha = 0.05$ level, there is significant but negative correlation between the participants' total ratings and their own scores on the written assessment questions (indicated by the * in Table 5.39), which suggests that those teachers who scored relatively lower on the assessment actually felt they have learned from the professional development activities at a higher level, compared with teachers who scored higher did. Content-specific professional development activities have greater impact on teachers' who have relatively weaker mathematical background.

Table 5.39 Correlation between teachers scores and ratings

		Total score	Total rating
Total score	Pearson Correlation	1	-0.277(*)
	Sig. (2-tailed)		0.018
	N	72	72
Total rating	Pearson Correlation	-0.277(*)	1
	Sig. (2-tailed)	.018	
	N	72	72

Major factors that shape teacher knowledge and preferences

At three points during each interview, I asked the following question to the participant:

> Which of the following factors have had the biggest influence on your knowledge and conceptions regarding what we have just discussed?
>
> 1. High school mathematic study
> 2. Advanced mathematics preparation
> 3. Mathematics education method preparation
> 4. Professional development workshops
> 5. Collaborating with/assisting/learning from colleagues
> 6. Using and appraising algebra textbooks and other instructional materials
> 7. Using manipulatives and technology
> 8. Dealing with student questions and conceptions
> 9. Implementing state standards / preparing for state exams

The question was first asked right after the participants answered the first interview question about their general conceptions of and preferences on teaching equation solving. After having gone through with the participants all five methods for solving linear equations and they have completed rating those methods by the eight measures, I asked the question for the second time. Each participant made quite consistent choices in answering the question for these two times. The common combinations of factors they gave include: 4 and 5; 4, 5, and 7; 5 and 8; 5, 7, and 8. Factor 5 (Collaborating with, assisting, or learning from colleagues) was the most common factor, mentioned for a total of 11 times. Seven of the participants each mentioned factor 7 (Using manipulatives and technology) once, and factor 4 (Professional development workshops) was mentioned six times. Tom explained why Factors 7, 4, and 5 have been most influential to him:

> When I was in school we didn't have all these tools. We didn't use manipulatives very often, we didn't use graphing calculators. I remember in our algebra class we graphed the equations on the Apple II computer which didn't have monitors back then. The thing is, when I came into teaching, the only method I knew were how to do these things by hand. So with the advent of technology, with the advent of the manipulatives… so, basically, learning from other colleagues, getting information from other teachers, either be one-on-one talking, going to a workshop seeing what they present... those are the most helpful for me.

As to Factor 8 (Dealing with student questions and conceptions), it was mentioned by five participants for a total of eight times. Pam and Yvonne offered two different kinds of accounts since they are working with students in a magnet program and in a low-achieving class, respectively:

> As a learner myself, there are certain ways I did things, and I may not always get the one given by the teacher. You build up the way you solve problems in you

> head, you have your kids questioning you, say "hey can I do this too?." You really think "Oh, well, you know, there are really a lot of ways to do this," so you have to work it out. So dealing with student questions and conceptions is definitely important.
>
> You have to understand I don't teach upper kids, I teach at-risk kinds…If I have Pre-AP classes, you can do anything you want. For those slower learners, you can't. You have to slow it down for them, or teach different ways that they can understand, like yesterday I had a kid came up asking a question about $x^3 + x^2 + x$… they couldn't tell why we can factor x from there… So I had to find a real-life example to explain to them. They have a lot of questions, I have to answer them. If they have a misconception or something, I have to address it, just like I did yesterday.

In contrast, none of the participants ever mentioned Factor 3 (Mathematics education method preparation). Amy's account may be representative in explaining why that is the case (and also why she instead chose Factor 4 and Factor 5):

> Looking at college, if you plan to be a teacher, a lot of what we are learning is in a way in a vacuum… we are just learning for our own sake, to say "OK, I have mastered this concept, I understand how to do this." When we get out into our job of teaching, its now not only about what *we* know about the subject, its about the teaching of the subject… One thing I've found is, if I were to take a test now on the math that I know, it takes a lot longer now than it did 15 years ago when I was in college. Because now when I do the problems I ask myself OK, how would I explain it to a student? And that's hard to learn all by yourself in a course that you are taking. So what I've learned the most is the professional development workshops… now we have known how to do this, how are we going to teach it to our students? The same thing with collaboration with colleagues, I have someone to look at teaching in the first period, I've learned a couple of things with regard to how she teaches it, that I haven't thought of it for certain topics.

Factor 6, Using and appraising algebra textbooks and other instructional materials, was selected only once, by Mary:

> What the textbook can give you that you are not going to get out of university materials is …. it gives you a pattern that gives you the ability to look at a contextual type problem and allows you to see from a different point of view. It also allows you to see where you going to take this to the next. So I really do

> think that it helps you organize your thought. And without that organization, sometimes you fall apart because a teacher is only as good as they prepare, and that's the truth. I mean if I don't prepare well, my students are not going to learn well.

Among many possible reasons, Rene gave one explanation for why textbooks have not had influenced teachers' knowledge, which was tied to three interrelated phenomenon in her school: many minority and immigrant students have different language backgrounds, they do not do homework with textbooks, and it is a popular practice of teachers to not use officially adopted textbooks:

> I don't think textbooks do much…I never taught out of textbook… Why? Students have different language…they are terrified of textbooks. So if you try to use textbook, they shut you down immediately. It's a tough situation…
>
> The first year here I was told nobody in this school uses textbook. It's such a school…they lose them…you don't have time to find them. So I said forget them…just like everybody else, why should use textbooks…otherwise you just created a lot more work to do…nobody in his school uses…the only classes… Pre-Calc and Calculus use… not even in Algebra 2….they might have books in the room, but they don't teach out of it.
>
> The school district has its own approved textbook. People have like to have one set in their classroom. But students don't take anything from school home. They don't do work outside school, so there is no point give a book… no use….but they will never be successful in college ….

The situation described by Rene regarding the use of textbooks may not be obsolete. Six of the eight teacher interviewees stated that they went with instructional packages developed by the district, the school, or other institutions, with a partial combination of officially adopted textbooks, or completely left them out.

Jane, a teacher with 16 years of experiences, gave another reason: “When I’m looking at a textbook I know pretty much what I’m looking for, by this time of my career that probably hadn’t have that big influence on me…”

Factor 9 (Implementing state standards, preparing for state exams) were never mentioned by any of the participants as having major influence. Referring back to the survey question about the participants’ professional development activities, “preparing for district and state assessments” had the second highest ratings in terms of how useful the activity has been. An explanation that could potentially rationalize the participants’ contrasting evaluations of exam-preparation type of activities would be that, to the participants, there is fine distinction between the knowledge needed for preparing students for exams and the knowledge needed for teaching concepts and processes.

Near the end of the interview, the participants were asked this question again after they talked about their conceptions of the role of mathematical procedures and rules. Half of the participants picked Factor 2 (Advanced mathematics preparation) as having the key impact, in the sense that college mathematics studies helped them seeing the bigger picture and understand better why procedures are important and why they work. The other half chose Factor 8 as the answer. For them, working with students that have different knowledge, skills and attitudes in relation to procedures and rules, is a constant theme in their teaching practices.

Chapter 6 Conclusions, Implications, and Recommendations

This study focuses on (a) what secondary school algebra teachers utilize within and across the basic domains of mathematical knowledge when pondering problem situations that could arise in teaching solving algebraic equation and how the knowledge is used, and (b) how teachers' backgrounds and experiences may influence the status and use of their knowledge. This chapter first summarizes major findings related to the issues above, and then further reflects on the nature of mathematical knowledge for teaching. The chapter ends with discussions of (a) the implications of the findings both for improving and for assessing teachers' mathematical knowledge for teaching, (b) the limitations of the present study, and (c) recommendations for future research.

SUMMARY OF MAJOR FINDINGS

What Types of Knowledge Were Used

In this study, assessment questions were initially designed with the intention of uniquely targeting and assessing each of the three basic domains of mathematical knowledge that teachers may use in teaching equation solving: knowledge of the subject matter, knowledge of learner conceptions, and knowledge of didactic representations. The data collected did help to achieve the goal. One characteristic of these kinds of questions is that, to each question, there typically exists a correct answer well agreeable among graders or evaluators.

For those questions that involve teaching and learning contexts, such as evaluating and responding to student ideas, and selecting and sequencing topics in teaching, the participants' responses revealed increased complexity: teachers' subject matter knowledge may go into the background as the other two domains of mathematical knowledge become more explicit in and influential to teachers' thinking. Teachers may even utilize other types of knowledge that are not necessarily specific to equation solving or algebra, such as their knowledge of learners' general characteristics and development in studying mathematics, and their general pedagogical knowledge. On some occasions, when the questions become broader or more open-ended, some teachers may mainly rely on those general types of knowledge for reasoning and decision-making.

How Different Types of Knowledge Were Used and Related

Here, the question of *how* knowledge is used actually integrates two interwoven aspects: (a) *In which ways* is the knowledge used? (b) *How well* is the knowledge used? Below I distinguish among six types of situations.

Reasoning about the subject matter within mathematical contexts

Teachers apply their subject matter knowledge to answer those questions that are phrased as in purely mathematical contexts (e.g., the prerequisite knowledge and skills for a certain equation solving strategy, the constraints and applicability of a strategy or property, and the mathematical connections among methods and concepts). These kinds of questions may not be something that algebra students have to think about or concern themselves with, but these questions would be essential measures of the depth, breadth, and connectedness of teachers' subject matter knowledge.

A major finding in this type of knowledge use is algebra teachers' limited understanding of two central and intertwined entities in school algebra: the balancing method for solving equations (i.e., performing the same operations on both sides of the equation) and the notion of equivalent equations. A large proportion of teachers seem to believe that they would always generate equivalent equations (i.e., keep an equation "balanced") as long as they "do the same thing on both sides," without being fully aware of the fundamental distinction between the kind of transformations through which equivalence is guaranteed by the properties of equality and those more sophisticated transformations which may possibly yield non-equivalent equations.

Teachers' limited understanding of the balancing method possibly has its roots in two commonly used pedagogical strategies: (a) using real-life models and physical manipulatives, such as balance scales, to render intuitive convictions of the balancing method, and (b) signifying the four basic properties of equality with the concise rhetoric, "doing the same thing on both sides," or with the label, the *Golden Rule of Algebra*. Despite their contributions to student learning, such intuitive convictions and concise significations cannot fully display those "invisible" effects of transformations on an algebraic equation, especially changes affecting the domains of the variables.

Some teachers have a narrow understanding of the scope of equivalent equations. Even though the definition was explicitly provided (equations that have the same solution set), many of them still think of equivalent equations as always being parallel, having the same y-intercept, or even being identicial. These suggest that some teachers tend to interpret equations and equivalence from a graphical perspective and as if the equations must be in two variables. The equivalence between linear equations in one variable is neglected. To a certain extent, such a tendency might be related to the state's function-based algebra curriculum which stresses the study of equations in two variables and their multiple representations. Actually, four of the eight interviewees started talking about the graphs, tables, and dependent and independent variables of "linear equations" after I posed the first question about solving linear equations.

Teachers' inadequate understanding of the balancing procedure and the notion of equivalent equations has two immediate consequences:

1. Many teachers do not see the fundamental connections between the balancing method and the equivalence of equations. By definition, two equations are equivalent if they have the same solutions; hence, one can easily examine the equivalence of two equations by finding and comparing their solutions. But the balancing method actually provides another way for determining the equivalence between two linear equations: They are equivalent if one can find a basic transformation that links them together. The notion of equivalence is not merely about the last step solutions; it is also inherent in each step during the balancing process. This is the key idea that underlies Question 5.4. Except for the first pair of equations in 5.4, more than 40% of the participants were not able to figure out such connections (i.e., determining the equivalence of pairs 2) and 3) without solving them).

2. Such limited understanding can not provide solid support to teachers' reasoning about the graphical connections between a linear equation $f(x) = g(x)$ and its transformed relatives $f(x) + 3 = g(x) + 3$, $3f(x) = 3g(x)$, and $f(x) + 3x = g(x) + 3x$. Even though more than half of the teachers selected correct answers to the related multiple-choice questions, several of them became uncertain during the follow-up interviews, which suggests they may have made the correct choices based on intuition or a guess, rather than on rigorous reasoning. It was not until the end of the interviews that four of the participants were able to find convictions from the properties of equality.

A second major finding is that teachers are not used to investigating linear equations or functions in their general forms (e.g., $ax + b = 0$, $y = ax + b$, $ax + b = cx + d$, or $f(x) = g(x)$) and consequently are not familiar with some general graphical properties of lines or linear functions. For a linear function $f(x)$, teachers understand very well the relationship between the graphs of $f(x)$ and $f(x) + a$. However, it becomes much harder to sketch the graphs of $af(x)$ or $f(x) + ax$. During the interviews, all of the participants focused on the change in slope and understood that $3f(x)$ was steeper than $f(x)$, but no one considered the changes to the special points, such as the x-intercept or the y-intercept. As a result, none of them was sure about the exact position of the line $3f(x)$ relative to the line $f(x)$. A few participants tried with general symbolic forms (e.g., from $f(x) = mx + b$ to $3f(x) = 3(mx + b)$) or specific cases (e.g., $f(x) = 2x + 1$), but were not able to make further progress.

Subject matter reasoning in student learning contexts

Algebra teachers also apply their subject matter knowledge to answer questions in student learning contexts, such as evaluating and responding to student ideas or questions. Specifically, there are two types of situations:

1. Questions require teachers' sound subject matter knowledge and reasoning to generate a correct or reasonable answer, although the questions are posed as about student learning.

One example is Question 6.2 (responding to a student who thought the intersecting-the-line method did not work for equations with a constant on one side). Teachers who did not fully understand this method responded with, "Move the constant to the other side so it becomes 0," "you are right," or "I'm not sure." And those who answered "use your graphing calculator to try it" did not fully capture the key issue here: the student does not understand that y = constant also represents a line. So even with a calculator, the student may still not know what to do with the constant.

Another example is Question 6.4 (responding to a student who said that the intersecting-the-lines method and the graphical method for solving a system of linear equations are the same). A total of 48.6% of the participants had completely wrong answers or were not sure. This means that nearly half of the teachers were not aware of the difference between the two kinds of solutions (i.e., between a number, or the x-coordinate of the intersection, versus the intersection point, or an ordered pair). This is confirmed by the results of Question 6.5, for which some teachers determined that the solution to $f(x) + 3 = g(x) + 3$ was larger than the solution to $f(x) = g(x)$ because the intersection of $f(x) + 3$ and $g(x) + 3$ was three units higher than that of $f(x)$ and $g(x)$. This also confirms a previous finding from the case study done by Chazan, Larriva, and Sandow (1999).

2. Questions can be approached in different ways and may not have a unique correct answer. Teachers with stronger subject matter understanding would provide answers with higher quality.

For Question 3.6 (identifying a student's thinking behind the way he solved a quadratic equation and explaining to him why it does not work), an answer such as "the student was moving all variable terms to one side" or "the student was factoring" correctly describes what the student was *doing*, but it does not disclose completely what the student may be *thinking*. An understanding that the student was over-generalizing the zero-product property to one would explain better why the student had done all the steps.

Further, when teachers explained to the student why the strategy does not work, answers like "explain to the student the zero-product property" or "the equation has to be in $f(x) = 0$ form in order to be solved with factoring method" are valid, but they become substantial only when a teacher tries to further explain why $ab = 1$ does not imply $a = 1$ or $b = 1$. In the assessment, 72% of the participants who answered the first part of the question in a subtle way provided substantial answers to the second part of the question.

A second example is Question 3.2, in which teachers were asked to explain whether a trinomial like $x^2 + 4x + 6$ is "factorable." There is nothing wrong if a teacher claims that it is "not factorable" because none of the paired factors of 6 (1 and 6; 2 and 3; -1 and -6; and -2 and -3) would add up to 4. A more subtle issue is that the definition of factoring used here has an implicit condition: All the coefficients are integers (i.e., we are talking about factoring on the integer set ***Z***). Going beyond this set, the same trinomial would become "factorable" in later algebra study (e.g., in the real number set ***R***, by using the quadratic formula). For the longitudinal coherence of mathematics study, a teacher with profound subject matter understanding could emphasize that this trinomial is not factorable when we require all coefficients to be integers but that in later studies it will be factorable when the coefficients are no longer limited to integers. Such note could bring an invisible assumption from the background into the foreground and set the prelude for studies in the near future. In the assessment, among the teachers who claimed that the trinomial was not factorable, only 10.1% discussed the integer set or the constraint on the coefficients, while nearly three-quarters did not mention anything about the conditions.

Analyzing learner conceptions on the basis of subject matter understanding

Through the assessment, teachers identified a variety of student mistakes and difficulties in solving linear and quadratic equations. The instances of student mistakes made when using the balancing method that were most frequently mentioned by the teachers fall into the following three categories:

1. Problems with the negative sign, subtraction, and additive inverse.
2. Performing imbalanced operations on the two sides.
3. Combining unlike terms.

Such frequency of teacher-identified student mistakes could be used as a proxy for the actual frequency of mistakes made by students in classrooms. As shown in

Chapter 2, many instances of student mistakes and difficulties in the above three categories have been studied and documented by researchers.

Teachers reported the mistakes and difficulties by giving examples or general descriptions. It is not clear whether they would have reflected on these instances and attempted to synthesize some of them into more systematic or theoretical constructs, like the *didactic cut* (Filloy & Rojano, 1984), the *cognitive gap* (Herscovics & Linchevski, 1994), the *redistribution error* (Kieran, 1984), or the *procedural* and *structural* views of equations (Kieran, 1992). During the interviews, two of the eight teachers mentioned that some of their students liked to use the cover-up method when beginning to solve simple linear equations. But when asked whether elementary school students who were studying arithmetic would view equations in special ways, only one teacher (Pam) explicitly brought up an example of students' inappropriate notion, $5 - 2 = 3 \div 3 = 1$, and linked it to some students' inability in distinguishing between simplifying an expression and solving an equation. None of the other seven teachers felt students' understanding would deviate from the regular view of the equal sign as an indicator of balance or sameness. This is also confirmed by the result that many teachers did not realize there were certain types of equations that could not be solved with the undoing method because it treats an equation as a sequence of operations on the unknown that leads to a final numerical result.

In applying the quadratic formula, the top three types of student mistakes are as follows:

1. Problems with computations under the square root, particularly order of operations.

2. Problems with the negative signs and rules for integer operations involving the negative sign.

3. Problems with the ± sign.

These mistakes seem to be mostly related to students' insufficient knowledge and skills in arithmetic and have not been widely reported by research studies on algebra learning.

Teachers' knowledge of student conceptions could be developed, reinforced, and verified in their individual teaching experiences. Without attending to their daily practices, it is hard for us to determine fully the authenticity, generality, or quality of such knowledge. Nonetheless, we could at least evaluate the mathematical legitimacy of a teacher's claim that a certain student conception is incorrect.

In the study, some of the typical student mistakes identified by the teachers are questionable from a mathematical perspective. For example, several teachers wrote "dividing and multiplying before adding or subtracting" as one mistake. They may have equated mathematical routines that are convenient or efficient with mathematical rules that must be followed. Or, if the teachers do understand the differences, it may simply reflect a dilemma inherent in teaching mathematical routines: On the one hand, most of the interviewees believed that routines were not absolute, that there were exceptions or flexibilities in following the routines. On the other hand, as several interviewees stated, routines provide order, structure, and consistency, especially for beginning or lower level learners. Without routines to follow, the students may lose their direction. It is probably from this second perspective that some teachers considered deviations from rules as mistakes.

A few other teachers claimed that some students do not know “minus and negative are the same thing” (in the sense that $a - b = a + (-b)$). These teachers may have oversimplified the connection between the two operations that are both signified by the word “minus”: the binary subtraction operation and the unary operation of taking additive inverse. This misconception has a similar nature to a previously discussed teacher misunderstanding: that of understanding the solution to a linear equation in one variable versus the solution to a system of linear equations. Some teachers gave insufficient attention to the subtle differences between two concepts or the processes behind their surface level similarities or connections.

In summary, for knowledge use in this category, a profound subject matter understanding is a necessary, although not necessarily sufficient, condition for teachers to comprehend accurately and appropriately handle students’ misconceptions.

Reasoning about didactic representations on the basis of subject matter understanding

Three questions were designed to assess teachers’ knowledge of didactic representations involving manipulatives: using the balance scale or algebra tiles to solve linear equations and using algebra tiles to factor trinomials. A basic prerequisite for answering correctly a question in this category is that the teacher understands the mathematical processes. Beyond that, teachers’ experiences with the manipulatives played moderate to significant roles in questions involving algebra tiles.

In solving linear equations, teachers who had taught with algebra tiles did better than those who had not, but one-third of the teachers in the first group still thought an equation like $5x + 3 = 14$ could be solved with algebra tiles. This could be because they were not aware that algebra tiles can represent only integers, not fractions or decimals. Alternatively, they may not have solved the equation carefully or noticed that the solution was a fraction in the first place. In factoring trinomials, experience with algebra tiles played a more significant role: Nearly 80% of the teachers who had taught with algebra tiles provided correct answers, while two-thirds of those who had not gave completely wrong answers or no answers.

The case with the balance scale is a bit more complicated. While 61.1% of the teachers demonstrated how to solve the equation $2x + 1 = 5x + 7$ by automatically applying negative weights or using other strategies (such as balloons or weight underneath the balance), 18% of the teachers believed that it could not be solved with the balance scale because negative weights did not exist or that after moving all the weights from one side to the other, it was impossible for the physical balance still to be maintained.

This highlights a subtle issue related to using manipulatives for teaching mathematics, or teachers' knowledge of didactic representations: the discrepancies between purely mathematical procedures and their physical representations. The manipulatives are visual and hands-on. They are able to signify certain key features (quantity, dimension, sign, operations, etc.) of the mathematical concepts and processes being studied. Still, they lack the flexibility or generality of abstract mathematical symbols. If we insist on representing all concepts or procedures with physical media, constraints and limitations will soon surface, and at some point the representation may cause cognitive conflicts for some learners and teachers. For instance, some conflict might arise about whether negative weights exist in the real world, whether an algebra tile, *x*, possesses an arbitrary or a specific magnitude, or how a square-shaped tile could have a negative area.

Based on the above discussions, teachers' profound knowledge of didactic representations involving manipulatives could be distinguished at three levels:

1. Knowing how to use the manipulatives to solve typical problems shown in the instructional materials.

2. Knowing the mathematical limitations of the manipulatives, in terms of for which types of problems the manipulatives do not work and why.

3. Being aware of those potential conflicts that could be caused by the use of manipulatives among the mathematics, the real world, and learners' cognitions.

Related to the use of manipulatives, another kind of knowledge of didactic representation has been observed in this study: teachers' use of metaphors and analogies in teaching. During the interviews, several teachers described the kinds of metaphors they have used before in explaining concepts and procedures. For example, for the balancing method, the interviewees mentioned metaphors such as keeping physical balance (Tom), a scenario from a fun game (Jane), a real-life example of doing the same thing on both sides: "brush both sides of your hair in the morning" (Teresa), and an example about money (Yvonne). To illustrate the undoing method, Teresa talked about walking on a two-way street: If you go one way, then you have to go the opposite way to come back to the beginning point. Mary used the example of putting on clothes in the morning and taking off clothes before going to bed at night. For teaching the distributive property, Yvonne portrayed giving money to a group of people as equivalent to giving a certain portion to each person, and Teresa talked about taking one blue and one red tile out of three piles of tiles which all contained at least one blue and one red tile.

Similar to the use of manipulatives, using narrative, real-world metaphors, and real-world analogies could pull vivid images and authentic experiences into students' comprehension. But unlike those popularly used manipulatives, selecting and using metaphors are mostly teachers' personal choices; there are no standard or commonly accepted approaches or criteria for their use. A metaphor or an analogy might miss some key mathematical features of the concepts or procedures it is intended to represent, or introduce extra information that could distract learners from focusing on the key mathematical features. In some cases, the metaphors and analogies that teachers use are even mathematically flawed or incorrect (e.g., the ones used by Yvonne and Teresa for the distributive property).

Preferential uses of the three types of knowledge

Several of the open-ended questions received much more diversified types of answers from the teachers. Through the lens of the conceptual framework, these answers were categorized by the three domains of mathematical knowledge for teaching, and the frequencies were summarized to reflect the participants' overall preferences.

The results suggest that, when they have the opportunity to apply any of the three domains of knowledge, (a) individual teachers have their own preferences for the domains of knowledge on which they will rely; yet, (b) all the teachers, as a group, tend to rely more heavily on their knowledge of learner conceptions than on their knowledge of the subject matter or didactic representations.

For example, Question 1.1 asks about the two most important reasons why the balancing method is the most commonly taught method for solving linear equations. Fifty percent of the reasons provided by the teachers were oriented toward student learning and conceptions, and 29.2% explained the importance of the method by its mathematical characteristics. Twenty-five percent of the participants gave two reasons that were both oriented toward learner conceptions, and 12.5% gave two reasons that were both focused on the subject matter. The remainder of the participants provided reasons based on mixed domains of knowledge.

Similarly, when asked about the ways in which students could benefit from learning the undoing method, 50% of teachers' answers focused on student learning, and 34% emphasized the mathematical features of the method. Furthermore, 34.7% of the participants oriented both answers toward student learning, and 20.8% focused both on the mathematical subject matter.

Specifically, each of the responses oriented toward learner conceptions can be categorized as one of the following six types:

1. Making sense of concepts and procedures.

2. Making connections between the abstract and the concrete.

3. Relating concepts and procedures to real-life situations.

4. Applying a method or carrying out a procedure.

5. Student mistakes and difficulties.

6. Student thinking skills and problem solving skills.

These categories might be broad enough to cover different aspects of student learning related to solving equations. Whether they apply to teachers' mathematical knowledge for teaching other topics or content areas could be investigated through future studies.

At this level, it is not always easy to determine into which unique domain a teacher's statement and underlying reasoning would fall. For instance, some teachers claimed that the balancing method "helps students to understand the meaning of equal and how to keep equal" or that "the principle is easy for teacher to state and easy for students to remember and understand." Some others believed that the undoing method "sets a stage for studying functions and inverse functions" or that "it connects to elementary school problem solving skills." Such kind of answers revealed that the three domains of mathematical knowledge for teaching could be blended together in one way or another. For example, the mathematical connections and sequencing among a few topics could be exactly the way in which a teacher wants to present the topics in teaching, or it could be exactly the way in which the students have understood.

Increased use of general knowledge in pedagogical decision-making

Two patterns are observed from teachers' knowledge use in making pedagogical decisions:

1. Teachers may draw upon knowledge of multiple types of didactic representations for the same topic, and some of the representations entail general pedagogical strategies.

2. Teachers increasingly replied on their knowledge of students' general learning characteristics. This could be viewed as an extension of teachers' preferential use of their knowledge of learner conceptions, as discussed in the previous section.

After teachers supplied instances of student mistakes and difficulties in using the balancing method and the quadratic formula, they also provided strategies for helping the students to improve their understanding. All of the strategies were grouped into categories by similarity. The following five categories include the highest number of strategies supplied for both the balancing method and the quadratic formula:

1. Using visual aids, hands-on, or metaphor to explain.

2. Reminding students of the rules, or reviewing prior concepts and methods.

3. Emphasizing the meaning of key concepts, symbols, and properties.

4. Rewriting a term or equation into alternative forms, so that it is easier to handle.

5. Practicing more.

The strategies in the first four categories are diverse but still very specific to the student mistakes and difficulties. They could also be viewed as entailing teachers' general pedagogical knowledge (such as using visual representation and direct instruction, or clarifying the meanings of concepts). The fifth category, practicing more, is a very general pedagogical decision.

In some other occasions, knowledge of learners' general characteristics became the determinant in teachers' decision-making processes. For example, 48.6% of the teachers understood that the completing the square method was the key idea underlying the derivation of the quadratic formula. Nonetheless, among these teachers, only about one-third felt that it was necessary to demonstrate to their students how the quadratic formula was derived and at least ask the students to remember the main ideas. Teachers who did not expect their students to know anything about the derivation were more concerned with students' learning characteristics (such as "I don't want to confuse my students with too much information," "it is too intense (or complicated) for the students," "students are not prepared," or "students don't care").

In characterizing the strengths and limitations of using manipulatives to teach equation solving, a few teachers also used their general knowledge about, and experiences with, mathematics learners. For example, "Students are different learners," "for those kinesics type of learners or lower level kids, the manipulatives help them to see the pictures and see what's going on," "students enjoy using their hands a lot," and "working hands-on keeps them paying attention and helps them to retain information."

Similarly, in discussing their goals of teaching multiple strategies for solving equations, several teachers emphasized students' differentiated learning preferences, styles, and abilities. They believed that practicing the basic routines was more suitable or even necessary for low-achievement students. Other students could stay with one strategy that best suits them, or they could always select the optimal strategy from among several, depending on the problem situation.

How Teacher Knowledge and Teacher Characteristics are Related

Findings on the relationships between various components of teachers' mathematical knowledge for teaching equation solving are the results of (a) analyzing and comparing teachers' performance on the assessment instrument in relation to major variables in their backgrounds and experiences, and (b) summarizing patterns in their responses and accounts in the follow-up interviews. These investigations and findings align with researchers' call for studying mathematical knowledge for teaching by connecting teacher characteristics with their knowledge (e.g., Ball, Lubienski, & Mewborn, 2001) and also supplement previous studies that made remote links between teachers' backgrounds and student learning outcomes.

Factors that make a significant difference on teacher knowledge

Three factors are found to make significant differences in algebra teachers' mathematical knowledge for teaching equation solving that are demonstrated by their performance on the assessment: college major, course-taking in advanced mathematics, and course-teaching in school algebra.

1. College major and advanced mathematics course-taking

College major and advanced content course-taking have been used as proxies for teachers' knowledge in major studies on the relationships between teacher knowledge and teaching effectiveness (Darling-Hammond, 2000; Goldhaber & Brewer, 2000; Monk, 1994; Rowan, Chiang, & Miller, 1997). Overall, these studies have shown that (a) having a college major and (b) having taken around five advanced courses in the content area in which a teacher educates significantly impacts teachers' quality of instruction, which was typically measured by student achievement, or teacher performance on evaluations, or both. This dissertation study takes a closer at these connections by directly measuring teachers' mathematical knowledge for teaching algebraic equation solving and linking it to teachers' academic preparations and teaching experiences.

Some of the results from this study are basically consistent with the findings from those aforementioned studies but reveal more direct relationships: Teachers who have a college or graduate degree in mathematics do have significantly higher performance on the instrument that assesses mathematical knowledge for teaching equation solving than those who have non-mathematics majors (mean difference = 9.7, $p = 0.012$). And there is a linear correlation between a teacher's performance on the assessment (P) and with the number of advanced mathematics courses the teacher has taken (C, up to seven) with significant coefficients: $P = 39.28 + 3.44C$ ($p < 0.001$, $r^2 = 0.17$). The small coefficient of determination (0.17) suggests that other confounding factors (such as college major and the types of algebra courses a teacher has taught) also contribute to a teacher's performance.

There are not enough data to verify the previous results that the difference is no longer significant among those who took more than five mathematics courses (Monk, 1994).

Significant correlation was also found between teachers' college major and advanced mathematics course-taking: Mathematics majors tend to take a greater number of advanced mathematics courses than non-mathematics majors ($r = 0.569$, $p < 0.001$). Nonetheless, the number and titles of mathematics courses a teacher has taken provide richer information about the teacher's mathematics background. And a teacher who had a non-mathematics degree might have taken the same number of, or even more, mathematics courses than an average mathematics major. Therefore, mathematics course-taking is a more informative and precise indicator of a teacher's mathematics background.

2. School algebra course-teaching

Another factor that proved to make a significant difference in teachers' mathematical knowledge for teaching equation solving is the kind of school courses that a teacher has taught. Specific to algebraic equation solving, teachers who are most experienced in teaching either first-year algebra or more advanced algebra courses performed better than those who are most experienced in pre-algebra courses ($p < 0.005$). This finding adds new and specific information to the knowledge base regarding the relationship between mathematics teachers' knowledge and their teaching experiences. It is not certain, though, whether the types or levels of courses with which a teacher has the most experience are confounded with the teacher's mathematics background (e.g., teachers with stronger mathematics backgrounds might get more opportunities to teach more advanced algebra courses).

Factors that do not make a significant difference on teacher knowledge

Two factors did not make a significant difference on a teachers' mathematical knowledge for teaching equation solving: the teacher's course-taking in mathematics education and how long the teacher has taught school algebra.

1. Course taking in mathematics education

Several early studies have shown that education coursework, including subject-specific methods courses, has positive impact on teaching performance or student achievement (Ferguson & Womack, 1993; Guyton & Farokhi, 1987; Monk, 1994). This study has a different, but not necessarily contradictory, finding: The number of mathematics education courses that teachers have taken (mathematics education methods, the psychology of mathematics learning, assessment in mathematics education, etc.) does not predict the strength of their knowledge for teaching equation solving, even though those who have taken more mathematics education courses tend also to have taken more mathematics courses. The reason could be twofold: (a) There is less agreement on the content for mathematics education courses across mathematics teacher education programs and institutions than on the content for mathematics courses, and (b) partially because of such variation, mathematics education courses may not be detailed enough to cover all the specifics of teaching and learning algebraic equation solving.

During the follow-up interviews, none of the eight teachers ever indicated mathematics education courses as having a major influence on their knowledge and decision-making.

2. Years of algebra teaching

Another finding in this category is that how long a teacher has taught algebra does not make a significant difference on his or her knowledge for teaching equation solving, despite the result that teachers who have taught algebra for six to nine years have higher average scores than their more novice (less than six years) or more experienced (more than nine years) colleagues. This would make sense if we consider the fact that when teachers are grouped by number of years of algebra teaching, each group of teachers would have mixed mathematics backgrounds and levels of course-teaching (i.e., within-group variance is likely to be larger than between-group variance). When there is larger amount of data collected from teachers, we could make more specific comparisons between certain teacher groups, e.g., those who are in their first year of teaching versus those have taught five years or longer.

Factors that need to be further studied

Three other factors, as well as their relationships to teacher knowledge, were explored in the study: (a) algebra textbook use, (b) prior experiences with teaching an algebra method or using manipulatives, and (c) participation in algebra-related professional development opportunities. The analyses have not shown consistent patterns in the role they play. Thus, these factors merit further investigation in future studies.

1. Algebra textbook use

Curriculum materials should play an important role in teacher learning and improvement of instruction (Ball & Cohen, 1996; Lappan & Rivette, 2004; Remillard, 1999; Remillard & Bryans, 2004). In this study, the participating teachers had very diverse experiences with algebra textbooks. Their uses spread across the 11 textbooks listed in the questionnaire, and many teachers also cited using other types of textbooks or curriculum materials, such as those designed by research and development institutions. Such diversity in textbook use, plus the small sample size of participants, made it hard to analyze the relationship between teacher knowledge and textbook use.

The current list of officially adopted algebra textbooks in Texas has been in existence since 1998. The state is approaching the completion of a new textbook adoption cycle. Through talking to teachers about their textbook use during the interviews, I realized the existence of another layer of complexity: Many algebra teachers, and even their schools, actually do not use, or at least do not focus on, the algebra textbooks approved by the state or their own school districts. Such a reality made it less surprising when I noticed that seven of the eight teachers did not think textbooks had influenced their knowledge and expertise.

This situation differs from the findings reported by Bush (1986) on preservice mathematics teachers who perceived the use of school textbooks as one of the major factors that shaped their teaching decisions. Along with the growths in their teaching experiences, many teachers may start to develop what Ball and Cohen (1996) called “the idealization of professional autonomy” which “leads to the view that good teacher do not follow textbooks, but instead make their own curriculum.”(p. 6)

Nonetheless, I would not conclude that algebra textbooks are insignificant to teachers’ mathematical knowledge for teaching equation solving. During the interviews I sensed the influence of the state algebra curriculum standards when half of the interviewees started to talk about tables and graphs of equations right after I asked them about linear equation solving. In answering assessment question 5.4, which regarded equivalent equations, many teachers approached it from slope and *y*-intercept perspectives, which indicated their inclination for thinking about equations in two variables, rather than in one variable. These two scenarios are consistent with the research finding that the current algebra curriculum standards in Texas put priority on the study of functions (equations in two variables) and their multiple representations (Li & Zhao, 2005).

Therefore, one conclusion that could be drawn is that, to a certain extent, teacher knowledge is shaped by their state and local standards and curriculum. But, for various reasons, teachers themselves may not be fully aware of the influence of either curriculum standards or the textbooks they use.

2. Prior experiences with teaching a method

There are uncertain relationships between (a) the depth of teachers' knowledge of a certain method or certain manipulatives and (b) whether they have ever taught that method or used those manipulatives before. One type of situation is that instructors who had taught the intersecting-the-line method for solving linear equations and who had taught factoring trinomials with algebra tiles performed much better on related questions than those who had not. A different situation is that, although those who had taught solving linear equations with the undoing method and algebra tiles performed better on related items than those who had not. the majority of teachers in both groups still performed poorly.

3. Professional development experiences

In the study, teachers also evaluated the impact of various professional development activities on their knowledge and practice. Teachers' questionnaire ratings yielded a significant but negative correlation between teachers' performance and their average ratings of nine types of algebra-specific professional development activities. This means that teachers who had relatively weaker knowledge for teaching equation solving had stronger feelings of having benefited from those learning opportunities. In the interviews, teachers identified three factors as the major sources of their knowledge: "Collaborating with and learning from colleagues," "dealing with student conceptions and questions," and "using manipulatives and technologies." Professional development workshops follow right after these three. All four factors deserve more thorough scrutiny in future studies, in terms of their individual effects on teacher learning and their interrelationships.

CONCLUSIONS

In resolving problem situations related to the teaching and learning of algebraic equation solving, secondary school algebra teachers have utilized, within and across, three basic knowledge domains: knowledge of the mathematical subject matter, knowledge of learners' conceptions, and knowledge of didactic representations. Beyond those, teachers have also made references to two other domains of knowledge: mathematics learners' general characteristics and general pedagogical strategies.

The study reveals three topic areas in equation solving in which teachers' mathematical subject matter understanding should be strengthened: the balancing method, the concept of equivalent equations, and the properties of linear equations in their general forms. When resolving problems situated in learning and teaching contexts on the given assessments, teachers provided a wide range of instances of student misconceptions and difficulties in learning how to solve linear and quadratic equations, as well as a variety of strategies for helping students, correspondingly, to improve their understanding. Teachers' subject matter knowledge still played a central or prerequisite role in reasoning and decision-making in these specific contexts.

When the problem contexts become broader or more general, teachers would draw from across the three basic domains of mathematical knowledge for teaching based on their individual preferences. Overall, though, teachers tend to rely more heavily upon their knowledge of students' specific or general learning characteristics.

Individual teachers have varied conceptions of the basic algorithms and alternative strategies for solving linear equations, in terms of their features, roles, differences, and connections. Teachers also have distinct expectations for and approaches to teaching multiple strategies for equation solving. Overall, student learning characteristics are determinant in teachers' conceptions and expectations.

Collectively, teachers have focused and balanced conceptions of the strengths and limitations of mathematical routines and rules.

Statistical analyses suggest that teachers who have mathematics majors, have taken six or more advanced mathematics courses, or have the most experience in teaching first-year algebra or more advanced algebra courses demonstrated a significantly higher level of mathematical knowledge for teaching than their counterparts. However, teachers' course-taking in mathematics education or their number of years of algebra teaching does not have a significant impact on their mathematical knowledge for teaching equation solving. Results are unclear or inconsistent about the role of three other factors in teachers' experiences: use of algebra textbooks; prior experience with teaching a method or using a manipulative; and participation in professional development activities. Teachers also rated two other types of experiences as highly influential on their knowledge growth: (a) collaborating with and learning from colleagues and (b) dealing with student conceptions and questions.

Reflecting on the Notion of Mathematical Knowledge for Teaching

During the data analysis and summary of findings, deeper or broader realizations surrounding the notion of mathematical knowledge for teaching emerged. This notion has been approached by researchers mainly from two directions: (a) defining subcategories of such knowledge and (b) specifying the typical teaching tasks for which such knowledge is likely to be used. These two aspects are actually inseparable, and the relationships between teachers' knowledge use and contextual situations are never simple. If our goal is to reasonably evaluate teachers' use of mathematical knowledge in practice and ultimately to improve the quality of their knowledge and its use, we cannot avoid the epistemological issue of justification: What are the sources of warrants for teacher knowledge and its use in contexts, and what is the nature of each source?

Two Lenses for Characterizations

Researchers have characterized mathematical knowledge for teaching mainly through two lenses: The first lens consists of identifying and defining the basic sub-categories of such knowledge. All five perspectives reviewed in Chapter 2, as well as the conceptual framework I developed for this study, have defined knowledge categories in various ways. A second lens includes specifying the core teaching tasks or situations in which teachers may likely draw upon their mathematical knowledge. These tasks or situations have been discussed by researchers such as Ball and Bass (2003), and Ferrini-Mundy, Burrill, Floden, and Sandow (2003).

The categorical lenses help to clarify the basic components of mathematical knowledge for teaching, how they are related, and how they build. They allow for patterns and commonalities in teacher thinking and reasoning possibly to be observed, described, and synthesized. Meanwhile, researchers have noted that "any categorization of teacher knowledge and beliefs is somewhat arbitrary. There is no single system for characterizing the organization of teachers' knowledge" (Borko & Putnam, 1996, p.675), and the categories of teacher knowledge within a particular system are not discrete entities, and boundaries between them are necessarily blurred (Marks, 1990). In reality, "possessing a body of such bundled knowledge may not always equip the teacher with the flexibility needed to manage the complexity of practices" (Ball, Lubienski, & Mewborn, 2001, p. 453). This was likely one of the major motivations for the contextual lenses to be introduced.

By studying knowledge use in contexts, we can find out not only teachers' knowledge of the patterns and predicabilities in student thinking, and of common approaches to developing mathematical ideas, but also teachers' thinking and reasoning involved in handling novel, unpredicted situations. This, in turn, could produce information and data that help researchers to reify, modify, relate, and contrast the categorical components of mathematical knowledge for teaching.

These two lenses were previously integrated in research studies in which researchers either used findings from the contextual analysis to inform the framing and refining of categorical constructs (Ball, Bass, Hill, & Schilling, 2005; Ball, Bass, Hill, & Thames, 2006) or outlined the teaching contexts in which assessment items designed by knowledge categories could be situated (Ferrini-Mundy, Burrill, Floden, & Sandow, 2003; Ferrini-Mundy, McCrory, & Senk, 2006; Floden & McCrory, 2007).

Even though the teaching and learning contexts did not form an explicit dimension of my conceptual framework, they were considered and integrated in the assessment instrument design. The complexities of the interactions between teachers and contexts became even more apparent to me when I analyzed the data and summarized the findings.

Knowledge Use in Contexts

Three basic patterns

Below, I discuss three basic patterns that I observed regarding the relationship between teachers' uses of knowledge and the contexts:

1. Teachers draw upon different types of knowledge when reasoning about a similar issue in different contexts.

The nature and scope of the contextual situations make a difference in teachers' use of knowledge and reasoning. In answering the assessment questions regarding solving linear equations with balance scales or algebra tiles, teachers mainly used their knowledge of the solving processes and solutions, as well as their knowledge of the limitations of these manipulatives, in terms of what they could or could not represent. In the follow-up interviews, teachers were asked a related, but much broader, question: What are the strengths and limitations of using manipulatives such as balance scales or algebra tiles to teach linear equation solving? Although some of the specific limitations that teachers indicated in the assessment were revisited again by several of the interviewees, most interviewees mainly emphasized the features of manipulatives from student learning perspectives (e.g., "students see what's going on," "students enjoy using their hands a lot," some students "find it confusing"); from representational perspectives (e.g., "the manipulatives are visual and concrete," students may be "overloaded by the extra information"); or based on other kinds of considerations (e.g., manipulatives are only used at school and students cannot bring it back home; the ultimate goal for learning is to go beyond physical manipulations and do higher-level thinking, so manipulatives play a limited role).

To further illustrate the role of contexts, I adapted assessment question 6.5 to situations of various types and specificities, and designed eight new questions (see the list below) across three types of contexts: mathematical, learner-oriented, and pedagogical. Although the mathematical core stays the same, the kinds of knowledge to which teachers may resort in answering each of these questions will likely vary.

Mathematical contexts:

M1. Describe the connections between the solution to the linear equation $3x + 5 = 11x - 7$ and the solution to the related system of linear equations $\begin{cases} y = 3x + 5 \\ y = 11x - 7 \end{cases}$.

M2. Describe the connections between the solution to a linear equation $ax + b = cx + d$ and the solution to the related system of linear equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$ (a, b, c, and d are real numbers).

M3. Describe the connections between the solution to a linear equation $f(x) = g(x)$ and the solution to the related system of linear equations $\begin{cases} y = f(x) \\ y = g(x) \end{cases}$ ($f(x)$ and $g(x)$ are linear expressions with real number coefficients).

Learner-oriented contexts:

L1. What would be the major pieces of evidence that a student in your first-year algebra class has fully understood the connections between the solution to a linear equation such as $3x + 5 = 11x - 7$ and the solution to the related system of linear equations $\begin{cases} y = 3x + 5 \\ y = 11x - 7 \end{cases}$?

L2. The students in your 9th grade Algebra 1 class have studied how to solve linear equations and systems of linear equations, but some of them do not understand the connections between the solution to a linear equation such as

$3x + 5 = 11x - 7$ and the solution to the related system of linear equations $\begin{cases} y = 3x + 5 \\ y = 11x - 7 \end{cases}$. What might be the main causes of such difficulty?

Pedagogical contexts:

P1. By using graphing calculators, how would you help your 9th grade Algebra 1 class understand the connections between the solution to a linear equation $ax + b = cx + d$ and the solution to the related system of linear equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$?

P2. What real-world examples would you show to your 9th grade Algebra 1 students to help them understand the connections between the solution to a linear equation $ax + b = cx + d$ and the solution to the related system of linear equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$?

P3. What kinds of assessment questions would you give to your 9th grade Algebra 1 students to evaluate their understanding of the connections between the solution to a linear equation $ax + b = cx + d$ and the solution to the related system of linear equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$?

2. In similar contexts, different teachers would draw upon different types of knowledge and, hence, make different instructional decisions.

In responding to assessment question 1.2, teachers provided seven categories of strategies for helping students improve their understanding and use of the balancing method. When it comes down to even one specific type of student misconception (e.g., regarding the negative sign, subtraction, and additive inverse), all seven categories of strategies still apply. This implies that, when handling similar student learning situations with similar understanding of the mathematical subject matter, teachers would still draw upon, in different ways, their own knowledge of student learning (e.g., visual and hands-on approaches, lack of conceptual understanding or basic skills); knowledge of didactic representations (e.g., using manipulatives, teaching by direct telling versus going through each step); or even knowledge of general pedagogy (such as drill and practice).

Even and Tirosh (2002) provided a case study of three 7th grade teachers, Benny, Gilah, and Batia, who were teaching algebra expressions from the same textbook. Benny was unaware of students' tendency to conjoin or "finish" open expressions (e.g., writing the expression $3x + 4$ into $7x$), so he wrote a rule for simplifying expressions on the board right after he posed the question, "What does $3m + 2 + 2m$ equal?" He was surprised by a student's claim that $5m + 2$ was $7m$ and did not understand the reasoning behind such a claim. As a result, he repeated the rule again and gave more examples. In contrast, Gilah was aware of this student tendency and considered it the main obstacle in teaching how to simplify algebraic expressions. She also believed that differentiating between the notions of like and unlike terms should precede simplifying expressions. So, she planned a comprehensive activity that focused on the two notions and then on collecting like terms. From the researchers' perspective, Gilah "seems to emphasize procedure knowledge only, with no explicit consideration of other kinds of knowledge nor of classroom culture" (p. 229). Batia explicitly indicated her awareness of the student misconception in her lesson plan. When she faced the situation in the classroom, she used a rich repertoire of strategies, which was characterized by short and quick teacher-student exchanges. Her understanding of students' mathematical learning enabled her to make fast and relevant responses to students, but she did not allow them opportunities to interact among themselves.

The case study illustrates teachers' varied uses of knowledge and consequent decision-making in similar contexts. The absence of awareness and understanding of the students' tendency to conjoin open expressions led Benny to a completely rule-based, "top-down" approach to teaching. Both Gilah and Batia were knowledgeable about the misconception, but they designed their lessons differently, based on other immediate knowledge: Gilah drew upon her understanding that the notions of like and unlike terms, and the procedure of collecting like terms are important foundations for simplifying expressions. Her approach was to build up higher-level conceptions from their foundations. Batia focused on the specific operations involved in the expressions as well as the order of operations, so her didactic strategies were analytical and focues on the structure of the expression itself. The strategies used by these three teachers correspond to some of those that teachers in my study used: (a) emphasize the rule, (b) emphasize basic conceptions and methods, and (c) analyze the process step-by-step.

3. In more general and broader contexts, teachers demonstrate preferential use of knowledge of student learning. They also tend to rely more on their knowledge of students' general characteristics as well as their general pedagogical knowledge.

These patterns have been reported and discussed in this study's data analysis and summary of findings. When the contextual information is not very specific, teachers tend to base their reasoning and decision-making on their knowledge, beliefs, and experiences related to the students they have taught (particularly, how students are different in their abilities, learning styles, and preferences; how a certain concept or method could be easy or hard for a certain type of learner: or what pedagogical strategies would work better for certain learners). Later in this section I will discuss this phenomenon more thoroughly.

A conceptual map

With the research instruments and data I was able to measure and identify the three basic types of mathematical knowledge for teaching equation solving, which partially evidenced the construct validity of the study. However, in more sophisticated contexts, teachers have drawn upon more than one types of knowledge and demonstrated different preferences. Such complexity could not be well reflected through the existing conceptual framework. To provide a better representation of teachers' uses of knowledge in contexts, I expanded the three domains of mathematical knowledge for teaching into a conceptual map (Figure 6.1), which illustrates various types of knowledge and beliefs, their potential inter-relationships, and their connections to various contexts.

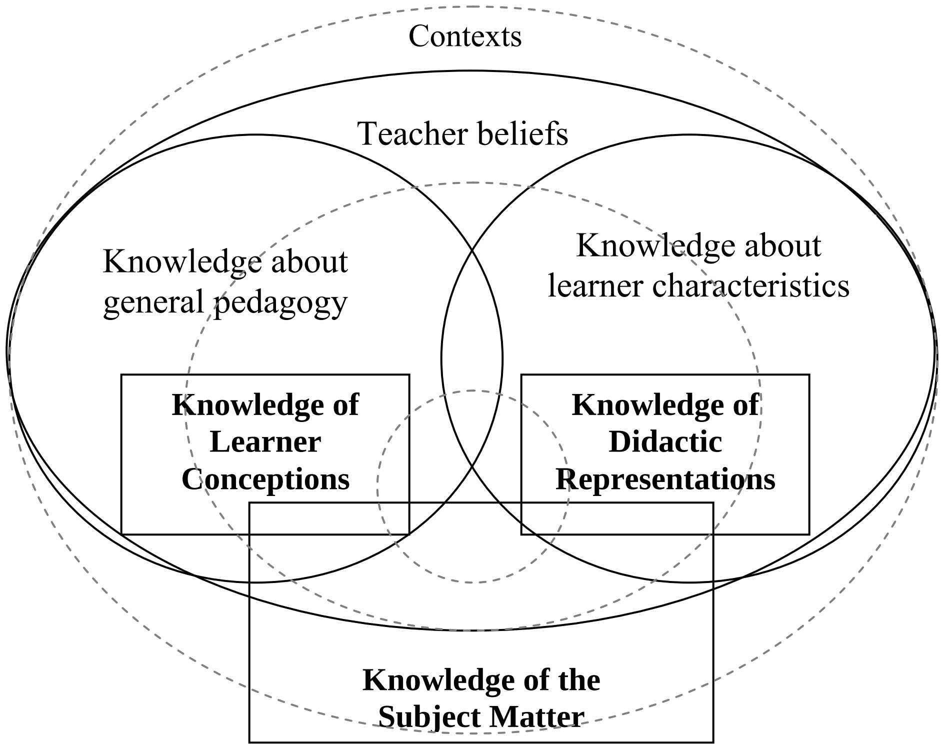


Figure 6.1 A conceptual map for teachers' knowledge use in contexts

At the core are the three domains of mathematical knowledge. Knowledge of learner conceptions and knowledge of didactic representations are based on, as well as connected by, teachers' knowledge of the subject matter. Meanwhile, knowledge of learner conceptions and knowledge of didactic representations are also a subset of teachers' knowledge of general student characteristics and knowledge of general pedagogy, respectively. However, this does not imply that developing the last two types of general knowledge would automatically guarantee the two previous types of knowledge.

The broadest cognitive terrain is teacher beliefs, which are based on individual experiences and, hence, may not be generalizable from one teacher to others.

The dashed circles represent contexts of various scopes. When the context is specific and focused, teachers generally use one or just a few focused types of knowledge, particularly the three domains of mathematical knowledge for teaching. The broader the context is, the more types of knowledge teachers could or would enact, and the more differential or preferential their decision-making could become.

Fennema and Franke (1992) built a model for examining teachers' knowledge as it occurs in the contexts of the classroom and described a general process for teacher learning and knowledge growth:

> A teacher must take his or her knowledge of mathematics, pedagogical procedures, and learners in general and apply that to the structuring of his or her classroom learning activities for specific learners. This knowledge is dynamic. Starting with the rudimentary knowledge of beginning teachers, it grows and matures as it interacts with specific learners in a classroom. (p. 160)

Considering the conceptual map in Figure 6.1, we may extend the above description to include the following: the interactions among (a) teachers' knowledge of mathematics, (b) the two types of general knowledge, and (c) knowledge of specific learners in classroom contexts would particularly shape teachers' knowledge of (a) how learners learn and understand specific topics and (b) what instruments and strategies to use in representing specific mathematics topics so that learners could understand. In this same process, teachers may also be able to deepen their own understanding of the mathematics topics they teach. In other words, the above interactions in contexts could build all three domains of teachers' mathematical knowledge for teaching.

In this study, evaluating teachers' mathematical knowledge for teaching and its use in various contexts encountered a few challenges. For example, when teachers provided different strategies for resolving the same type of student learning issue, it was not always easy to determine which strategy was more efficient or effective without having more detailed contextual information. Many of the teachers' responses and statements in the interviews seemed to be based more on their own beliefs or experiences than on their knowledge. Actually, researchers have noted that it is frequently the case that teachers treat their beliefs as knowledge (Grossman, Wilson, & Shulman, 1989). So, what can be called teacher knowledge, and how exactly is it different from teacher beliefs? How valid and reliable would a teacher's reasoning and decisions be if they were grounded on the teacher's beliefs? These questions lead to the discussion below about the justifications of mathematical knowledge for teaching.

Justifications of Knowledge Claims

By the classic definition found in Plato's *Theaetetus*, in order to count as knowledge, a statement must fulfill at least three criteria: be (a) justified, (b) true, and (c) believed. Therefore, one's knowledge can be considered a subset of his or her beliefs, and the major distinction between the two is whether one who makes a knowledge claim can find warrants to prove its validity. Feiman-Nemser and Floden (1986) stated that it does not follow "that everything a teacher believes or is willing to act on merits the label *knowledge*, although that view has some support" (p. 515). Strictly applying Plato's definition would certainly pose challenge to the notion of *teacher knowledge* that we have been commonly using, as well as all the related research and practice. Alternatively, if we use *knowledge* "as a generic name to describe a broad range of mental states of teachers that arise from their training, experience, and reflection and has little if any epistemological import", the consequence will be that we lose the basis "for deciding whether the knowledge of one teacher or researcher is better, more trustworthy, less troubled by error, or more resistant to objection and criticism than the knowledge of any other teacher or researcher." (Fenstermacher, 1994, p 34).

These above issues exceed the scope of this study. However, for the purpose of conceptualizing mathematical knowledge for teaching, it is helpful to ponder the various types or sources of justification for such knowledge. Below is my own summary:

1. Justification by mathematical criteria.

As we have seen in this study, many of the problem situations require teachers' profound understanding of the mathematics subject matter, regardless of whether the problems are in mathematical contexts or in teaching and learning contexts. Therefore, this type of teacher knowledge can be purely justified by its mathematical validity.

2. Justification by educational policies.

These policies include national education guidelines (such as No Child Left Behind); professional standards (such as National Board for Professional Teaching Standards (NBPTS) and Principles and Standards for School Mathematics by NCTM); state and local curriculum, teaching, assessment, and teacher certification standards; and accountability testing. Especially at the state and local levels, the standards are not all consistent. Although national and professional standards do have a system-wide impact, teacher knowledge for teaching would most likely be measured and justified against the state and local standards and contexts, and in some cases, by students' performance on high-stake testing.

3. Justification by theories or perspectives.

Sometimes we rely on educational, psychological, pedagogical, or epistemological theories or perspectives to justify teachers' knowledge for teaching, for example Piaget's theory of cognitive development (Inhelder & Piaget, 1958) and Bloom's Taxonomy of Educational Objectives (Bloom, 1956). Specific to mathematics learning, we have van Hiele's model of geometric reasoning (van Hiele, 1986), Dubinski's APOS theory (Dubinski, 1994), and theoretical frameworks such as the five strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001). Most of them are general theories about learning, and they certainly will not help to answer all specific questions in all situations. But what they can provide are benchmarks for examining whether teachers' knowledge and decision-making are aligned with learners' developmental understanding of certain mathematical concepts, topics, or subjects.

4. Justification by collective experiences.

These types of warrants include academic research data and findings on how students learn a certain subject (e.g., Donovan & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001), teachers' effective teaching experiences (e.g., teaching mathematics to minority students or students in urban schools), and expert mathematics educators' experiences and opinions. These researchers and experts disseminate their knowledge through publishing in academic journals and books, designing textbooks and other instructional materials for school students and their teachers, instructing teacher preparation courses, and organizing professional development activities.

Similar to the case of theoretical knowledge, researchers' and experts' knowledge is generalizable to a certain extent, and what it reveals about student learning and pedagogical strategies are mostly the patterns and regularities. It is not clear whether such knowledge can facilitates enough flexibility for teachers to handle novel or uncertain situations.

5. Justification by individual experiences and contexts.

This type of justification is two-sided: It could produce rich and detailed evidence within an individual teacher's practices and contexts, but it may also lack the generalizability of the previous four types of justification. Therefore, cross-referencing with other sources of justification would potentially make individual experiences and contexts a more rigorous justification.

These five types (and sources) of justification are the very basic ones. In daily teaching practices, teachers may count on different sources of justification or on a mixture of multiple sources. Hypothetically, those expert or effective teachers would be able to find a good balance or combination among these five sources. If this could prove to be true, it then becomes a mission of mathematics teacher preparation and professional development programs to help teachers learn how to look for and achieve that balance in applying their knowledge in practice.

IMPLICATIONS

Findings from this study have immediate implications for mathematics teacher preparation and professional development programs. Besides that, they provide hints for assessing teachers' mathematical knowledge for teaching.

Implications for Developing Mathematical Knowledge for Teaching

Although this study only involved practicing mathematics teachers, the findings have implications for both mathematics teacher preparation and professional development programs.

Majoring in mathematics or taking a sufficient amount of advanced mathematics courses increases the potential for prospective mathematics teachers to be both knowledgeable and effective in teaching. A profound understanding of the subject matter lays the foundation for teachers (a) to teach rigorous and coherent mathematics to their students, and (b) to develop subtle knowledge of learners' conceptions and effective representations of the content.

When certain weak areas exist in teachers' subject matter understanding, teachers' own practices (such as using a particular textbook, teaching a concept or method, or having taught over a long period of time) may not necessarily help to improve those areas. These weak areas must be specifically addressed and made explicit by mathematics content or education courses for preservice teachers and inservice teacher professional development activities.

Through teaching, teachers may develop a rich knowledge of student conceptions. But such knowledge tend to be informal and lacking in organization or coherence (Carpenter, Fennema, Peterson, & Carey, 1988). Professional development activities could help to upgrade teachers' empirical knowledge to more systematic and theoretical understanding (for example, recognizing elementary school students' underdeveloped notions of equalities and how these notions are linked to some of the strategies for solving equations). The CGI studies have shown that teachers' knowledge of student thinking can influence teaching and learning when it is both specific and organized (Fennema & Franke, 1992). The CGI model could be customized and implemented specifically for improving all aspects of teachers' mathematical knowledge for teaching. This customized model could allow for the following opportunities:

1. Teachers sharing their empirical knowledge of student conceptions.

2. Teachers and teacher educators examining, comparing, categorizing, and relating these student conceptions to expand them into more systematic knowledge.

3. Teachers possibly having deeper insights into the mathematical issues behind these student conceptions and the potential ways to help students improve their understanding.

4. Teachers applying this new knowledge in practice and possibly solidifying all three components of mathematical knowledge for teaching.

Some advantages of teachers learning through such a cycle or cycles include (a) the purpose and motivation that builds from seeking mathematical explanations and pedagogical resolutions and (b) the automatic grounding of the new knowledge in teachers' own experiences and student learning contexts.

Besides addressing the three basic domains of knowledge for teaching, teacher professional development programs should contrast and relate various sources of professional knowledge: national, state, and local policies and standards; research findings; expert experiences; individual experiences: and student needs. This would help teachers' individual and collective knowledge to become shared, more coherent, grounded, and balanced.

Implications for Assessing Mathematical Knowledge for Teaching

Corresponding to the two lenses through which teachers' mathematical knowledge for teaching has been conceptualized, its status can be measured in at least two dimensions: (a) the *quality* of such knowledge and its components, which are ideally measured quantitatively as high or low on certain numerical scales, and (b) teachers' *use* of such knowledge, its components, and related knowledge in contextualized situations, which may result in qualitatively described patterns and relationships.

Multiple-choice items, short open-ended questions, and task-based interview questions are direct and relatively efficient ways to assess the quality dimension of teachers' mathematical knowledge for teaching. One challenge for designing these kinds of questions is to make sure that the questions are much more focused and target a specific type of knowledge. Associated with that, it would require much effort to develop a set of robust answer keys and scoring rubrics. However, making the problem contexts as specific and detailed as possible may help to reduce the level of complexity in item design and potentially improve the reliability of the item.

Open-ended questions, concept mappings, interviews, and classroom observations are suitable for examining the use dimension, which includes how sensitive the teachers are to the contextual information, what types of knowledge are used, how they are used, and how reasoning and decisions are made. The scope of the context becomes a crucial factor for writing items that best elicit the desired information. In these contexts, teachers may draw upon their knowledge of students' general learning characteristics or general pedagogical knowledge, and it may not be easy to distinguish between teacher knowledge and beliefs. Multiple data sources and triangulations become particularly important.

Combining and correlating data and findings from the quality dimension and use dimension would yield a more complete account of the status of teachers' mathematical knowledge for teaching.

LIMITATIONS OF THE STUDY

In this study, the written assessment and interview questions were designed mainly based on the literature review and textbook survey. Some other potential sources for assessment questions were left out, for instance, state and local curriculum and assessment standards, observations of classroom teaching or analysis of existing video records of teaching. The mathematics topics covered in the assessment were the typical strategies for solving linear and quadratic equations. Other strategies (such as the cover-up method) and other types of school algebra equations (exponential, logarithmic, etc.) were not included.

The sampling was limited to a few school districts in Texas and, thus, not large enough to yield firm conclusions that are generalizable to larger scales or transferable to other policy and socioeconomic settings. Responses from the participants may not fully represent the diverse thinking and experiences of secondary mathematics teachers in various schools, districts, or states. The results and findings from the study should be considered only as preliminary and experimental; they should be validated or strengthened in future follow-up studies of larger scale.

Teachers' mathematical knowledge for teaching was measured through paper and pencil assessments and interviews; therefore, the data and results cannot reveal the full complexity of (a) classroom contexts, (b) the dynamic interactions among students and the teacher, or (c) teachers' reasoning and decision-making and how those relate to the contexts.

RECOMMENDATIONS FOR FUTURE RESEARCH

The current study is a first attempt to systematically study mathematics teachers' knowledge for teaching equation solving. The data and findings have produced brand new information for the research field. Nonetheless, there are aspects in the design to be improved, issues to be further clarified, and questions to be answered in future studies.

One immediate extension of this study would be to refine the assessment instruments and interview questions. Some of the open-ended items could be rewritten as multiple-choice questions, based on teachers' responses. Some other open-ended questions could be more contextualized so that there would be more solid criteria against which teachers' knowledge uses could be evaluated. And many interview questions could also be revised into task-based questions. Such revisions would likely increase the reliability of the instrument.

Another approach to refining the instrument is to analyze existing teaching videos or observe classroom teaching that is related to algebra equation solving. The empirical data could reveal mathematical issues and situations that require teachers' application of their mathematical knowledge, which would become sources of new and more valid assessment questions, particularly, those highlight teachers' knowledge of student conceptions and didactic representations.

With a more valid and reliable instrument, this study could be further extended in several ways. First of all, a longitudinal study could be conducted with inservice algebra teachers, so that a pre- and post-test comparison could be made to find out if there is any gain in teacher knowledge. If there is gain, the major contributing factors in teachers' teaching practice and professional development activities could be identified.

The revised instrument could also be customized for and administered to preservice mathematics teachers. The quality of these teachers' understanding could be measured and linked to their academic backgrounds. In addition, the data and findings from preservice teachers could be compared to those from the inservice teachers. The quantitative and qualitative differences would help us to understand better teachers' knowledge growth in relation to their backgrounds and experiences.

After having conducted research on small groups of inservice and preservice teachers, the study could be expanded to involve a larger sample at the district or state level. The results could then become more generalizable and, hence, have broader impact on policies and practices.

Beside providing sources of new assessment items, data from classroom teaching observations and analyses could be supported by pre- and post-teaching interviews, teaching logs, teacher journals and reflections, etc, and reveal a wealth of authentic and detailed information about the ways teachers draw upon various types of knowledge in predicable and novice situations, as well as the patterns in their reasoning and decision-making. For example, a specific research topic is: “How do algebra teachers select and use metaphors and analogies in teaching algebraic equation solving, and what types of knowledge are underlying the use?” A broader research topic could be “In teaching equation solving in various contexts, how do algebra teachers achieve balance between their knowledge of mathematical rigor and coherence and their knowledge of student learning characteristics?”

When this information and these patterns are linked to teachers’ performance in the assessment, it will advance our knowledge about the relationship between teacher knowledge and teachers’ reasoning and decision-making, and bring us one step closer to our ultimate goal which is to fully make sense of the impact of teachers’ mathematical knowledge for teaching on their teaching effectiveness, including their student learning outcomes.

Appendix 1 Academic Background Questionnaire

1. For how many years have you taught school mathematics?

   Middle school level ___________ High school level ___________

2. Select all the degrees you have earned or are in the process of earning, and specify the content areas:

   ☐ B.A./ B.S. Major ______________________________________
   Minor ______________________________________
   ☐ M.A./ M.S. Program ___________________________________
   ☐ Ph.D/ Ed.D Program ___________________________________
   ☐ Other: ____

3. In which school subjects and grade levels are you certified to teach?

   ☐ Mathematics Grade levels: ________________________
   ☐ Science Grade levels: ________________________
   ☐ Other: ____

4. Which of the following types of college or graduate courses have you taken? Check all that apply.

   Mathematics Courses
   ☐ Calculus
   ☐ Differential Equations and/or Multivariate Calculus
   ☐ Linear Algebra (e.g., vector spaces, matrices, dimensions, eigenvalues, eigenvectors)
   ☐ Abstract Algebra (e.g., group, field theory, ring theory; structuring integers, ideals)
   ☐ Number Theory and/or Discrete Mathematics
   ☐ Advanced Geometry and/or Topology
   ☐ Real and/or Complex Analysis
   ☐ Other: ____

Mathematics Education Courses

☐ Methods of teaching mathematics (planning, organizing and delivering math lessons, using math curriculum materials and manipulatives, etc.)

☐ Psychology of learning mathematics (how students learn math, common student errors or misconceptions in math, cognitive processes, etc.)

☐ Assessment in mathematics instruction (developing and using tests and other types of assessments)

☐ Other: ____

5. Which of the following algebra courses have you taught in the last five years? Check all that apply.

☐ 1) Pre-algebra ☐ 2) Remedial algebra

☐ 3) First year algebra ☐ 4) Second year algebra

☐ 5) Advanced algebra ☐ 6) Algebra in an integrated program

☐ 7) Other courses focused on algebra:

____

Among the algebra courses you have taught, you are most experienced with

____

6. Altogether, for how many years have you taught algebra courses? ______________

7. Which of the following textbooks or programs have you used to teach algebra in the last five years?

☐ 1) *Algebra: Integration, Applications, Connections / Glencoe Algebra*, Glencoe/McGraw-Hill.

☐ 2) *Algebra 1/Algebra 2*, Holt, Rinehart, Winston.

☐ 3) *Algebra 1/Algebra 2: An Integrated Approach*, McDougal Little/Houghton Mifflin.

☐ 4) *Algebra Tutor*, Carnegie Learning.

☐ 5) CMP Project: *Connected Mathematics*, Prentice Hall.

☐ 6) Core-Plus: *Contemporary Mathematics in Context: A Unified Approach,* Glencoe/McGraw-Hill.

☐ 7) *Discovering Algebra: An Investigative Approach*, Key Curriculum Press.

☐ 8) *Interactive Mathematics Program* (IMP), Key Curriculum Press.

☐ 9) *Mathematics in Context*, Holt, Rinehart & Winston.

☐ 10) *Algebra: Tools for a Changing World*, Prentice Hall.

☐ 11) Univ of Chicago School Math Project (UCSMP): *Algebra*, Scott Foresman/Addison Wesley /Prentice Hall.

☐ 12) Other:

Title	Publisher/Author
______________________________	______________
______________________________	______________

Among the algebra textbooks you have used, you are most experienced with ____________

8. Consider all the mathematics teacher professional development activities that you have participated in the last five years, to what extent have they been helpful to you in each of the following aspects?

	Not at all	Somewhat	Very much
1) How students learn algebra	☐	☐	☐
2) Algebra-related advanced mathematical knowledge	☐	☐	☐
3) NCTM and state standards specific to algebra	☐	☐	☐
4) Algebra textbooks and other curricular materials	☐	☐	☐
5) Methods for teaching algebra	☐	☐	☐
6) Using manipulative or technologies in teaching algebra	☐	☐	☐
7) Methods for assessing student learning in algebra	☐	☐	☐
8) Teaching algebra to students with diverse backgrounds	☐	☐	☐
9) Preparing students for district and state assessments	☐	☐	☐

Appendix 2 Written-response Assessment Items

1. In school algebra textbooks, one of the most commonly introduced methods for solving linear equations is **the balancing method**, i.e., performing the same operations on both sides of an equation based on the addition/subtraction and multiplication/division properties of equality.

1.1 Why is the balancing method the most commonly taught and used for solving linear equations? Please give two reasons that you believe are the most important.

1)

2)

1.2 In solving linear equations with the balancing method, what major types of mistakes (other than computational errors) or difficulties have you seen from students? Please list two different types of mistakes or difficulties, and correspondingly provide strategies for helping students to improve their understanding.

Major mistakes or difficulties	Strategies for improving student understanding
1)	
2)	

1.3 Below are two commonly used visual/hands-on models for teaching the balancing method. Indicate the extent to which you are familiar with each of them:

	Balance scales	**Algebra tiles**
I have taught with this model	☐	☐
I have seen or read about this model, but never taught with it	☐	☐
I know little about this model	☐	☐

1.4. Is it possible to solve the equation $2x + 1 = 5x + 7$ by drawing pictures of weights and balance scales? If yes, please demonstrate how it can be done. If not, please explain why not.

1.5 For each of the following equations, is it possible to use algebra tiles to demonstrate the solving process and accurately represent the solution? If not, please explain why.

	Yes	No	Explain why if you choose "No"
1) $5x + 3 = 14$	☐	☐	
2) $-3x + 5 = -10$	☐	☐	
3) $4x + 9 = 2x - 3$	☐	☐	

2. **The undoing method** (or, working-backward method) is sometime taught for solving linear equations. For example, the equation $2x + 5 = 11$ can be viewed as a sequence of two operations on x:

$$x \xrightarrow{\times 2} 2x \xrightarrow{+5} 11$$

To find the value of x, we begin with 11, undo the above two operations and get $x = 3$ at the end:

$$3 \xleftarrow{\div 2} 6 \xleftarrow{-5} 11$$

2.1 Have you ever taught the undoing method to your students? ☐ Yes ☐ No

2.2 On which kinds of **linear equations** can we directly apply the undoing method?

2.3 Does this method work for non-linear equations? What would be the characteristics of the kind of equations that can be solved directly by this method?

2.4 In which ways may students benefit from learning and using this method? Elaborate two major benefits:

1)

2)

2.5 In his homework, Joey tried to solve the equation $3x + 4 – 2x = 8$ with the undoing method:

Take 8, divide it by 2, add 4, then divide by 3.

What comments would you write down to help Joey correct his mistakes?

3. The following questions regard **the factoring method** for solving quadratic equations.

3.1 Which of the following algebra knowledge and skills are essential for students to understand the factoring method? Check all that apply.

- ☐ **A.** Combining like terms in an expression
- ☐ **B.** Multiplying two binomials
- ☐ **C**. The distributive property
- ☐ **D.** The zero-product property
- ☐ **E.** Solving linear equations of the form $ax + b = 0$

3.2 An algebra teacher says the equation $x^2 + 4x + 6 = 0$ cannot be solved with the factoring method because the trinomial $x^2 + 4x + 6$ is "not factorable". Do you agree?
If yes, how would you explain to your students what exactly it means that a trinomial is "not factorable"? If no, why not?

3.3 Have you ever taught your students how to factor a trinomial with algebra tiles? ☐ Yes ☐ No

3.4 For each of the following two quadratic equations, is it possible to use algebra tiles (or drawing their pictures) to solve it? If yes, please show how. If not, please elaborate why not.

1) $x^2 - 5x + 4 = 0$

2) $x^2 + 3x - 4 = 0$

3.5 In solving the equation $2x^2 - 5x - 3 = 0$, Karen factors it into $(2x + 1)(x - 3) = 0$. She then asks,
"*is this the only way of factoring it? How do we know?*"

1) How would you respond her?

2) Tony claims he did find a different way of factoring: $2(x + \frac{1}{2})(x - 3) = 0$. Is this valid? How would you respond to him?

3.6 Below is how Mark solved the quadratic equation $x^2 - 5x - 1 = 0$:

$$x^2 - 5x = 1$$
$$x(x - 5) = 1$$
$$x = 1 \text{ or } x - 5 = 1$$
$$x = 1 \text{ or } x = 6$$

1) What might he be thinking when he decided to solve the equation in that way?

2) Mark doesn't understand why this method won't work. How would you explain to him?

4. The following questions regard **the quadratic formula** for solving quadratic equations.

4.1 Through what major methods or strategies is the quadratic formula derived?

4.2 If you were to teach the quadratic formula, which of the following approaches would you prefer?

- ☐ **A**. Demonstrate how the formula is derived, expect students to understand and remember each step.
- ☐ **B**. Demonstrate how the formula is derived, expect students to remember the main ideas only.
- ☐ **C**. Explain the main ideas behind the formula, expect students to remember and use the formula only.
- ☐ **D**. Introduce and use the formula directly, explain how the formula is derived in late chapters.
- ☐ **E**. Introduce and use the formula directly, without ever explain where the formula comes from.
- ☐ **F**. Other:

___

Please elaborate why you prefer the approach you selected above:

4.3 List the two most typical mistakes that students make (or major difficulties they have) in learning and using the quadratic formula. For each case, please briefly explain what strategies may be most effective in helping students to improve their understanding.

	Most typical mistakes or difficulties	Strategies for improving their understanding
1)		
2)		

5. Some algebra textbooks introduce the concept of **equivalent equations** and define it as *two equations with the same solutions*.

5.1 Have you ever taught this concept to your students? ☐ Yes ☐ No

5.2 Does each of the following transformations on an equation **always** generate an equivalent equation? Give an example if you choose "No".

	Yes	No	Given an example if you choose "No"
1) Adding $\sqrt{x}$ on both sides	☐	☐	
2) Multiplying the two sides by $(x + 5)$	☐	☐	
3) Squaring both sides	☐	☐	
4) Taking square roots on both sides	☐	☐	

5.3 If $ax + b = 0$ and $cx + d = 0$ are two **equivalent** but **different** linear equations, what can we say about the two corresponding lines $y = ax + b$ and $y = cx + d$? Please determine the truth of each of the following statements:

	Always true	Possibly true	Impossible
1) These two lines are identical	☐	☐	☐
2) These two lines are parallel	☐	☐	☐
3) These two lines have the same *x*-intercept	☐	☐	☐
4) These two lines have the same *y*-intercept	☐	☐	☐
5) These two lines are perpendicular	☐	☐	☐

5.4 For each of the following pairs of linear equations, determine whether the two equations are equivalent or not, **without actually solving the equations.** Please explain your reasoning.

Pair of equations	Are they equivalent?	Your reasoning (without solving the equations)?
1) $3x - 4 = 16$ and $3x - 7 = 13$	☐ Yes ☐ No	
2) $2x + 8 = 4x - 15$ and $3x + 9 = 4x - 14$	☐ Yes ☐ No	
3) $2x - 4 = 3x + 16$ and $4x - 7 = 6x + 32$	☐ Yes ☐ No	

5.5 In learning solving linear equations, how much would students benefit from doing exercises like 5.4?

☐ Very much ☐ Somewhat ☐ Not much ☐ It depends

Please elaborate your thinking:

6. Some algebra textbooks introduce the following function-based method for solving a linear equation $f(x) = g(x)$:

Graph the linear expressions on the two sides simultaneously as two linear functions, $y = f(x)$ and $y = g(x)$, then determine their intersection (a, b). The x-coordinate of the intersection, a, is the solution to the original equation.

Figure 1 shows an example: to solve $3x + 5 = -4x - 2$, we graph the two linear functions $y = 3x + 5$ and $y = -4x - 2$, and find their intersection $(-1, 2)$. Then -1 is the solution to the original equation.

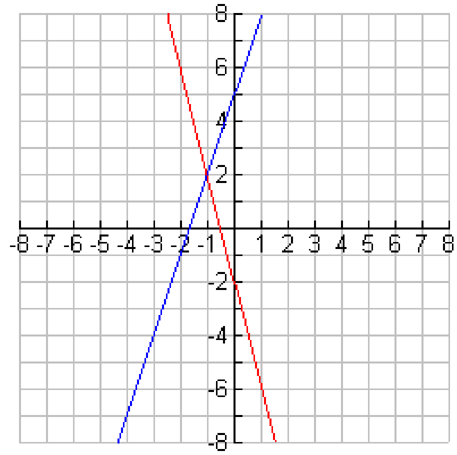


Figure 1

6.1 Have you ever taught this function-based method to your students? ☐ Yes ☐ No

6.2 Danny believes that the above method works for a linear equation only when the unknown variable x appears on both sides, i.e., it won't work if one side of the equation is a constant (for instance, $2x + 9 = 8$, or $5x - 13 = 0$).

What would you say to him?

6.4 Besides linear equations, for what other types of equations would this method also work?

6.4 Emily has learned how to use the above method to solve linear equations like $3x + 5 = -4x - 2$. Now she's learning the graphical method for solving a linear systems such as $\begin{cases} y = 3x + 5 \\ y = -4x - 2 \end{cases}$: graph the two lines $y = 3x + 5$ and $y = -4x - 2$, then find the intersection. She is happy to find out the connection: "these two methods are actually the same and they give the same solutions!"

Is her conclusion valid? What would you say to her?

6.5 $f(x)$ and $g(x)$ are two given linear functions. Figure 2 shows the function-based method for solving the equation $f(x) = g(x)$. Suppose the solution is $x = a$.

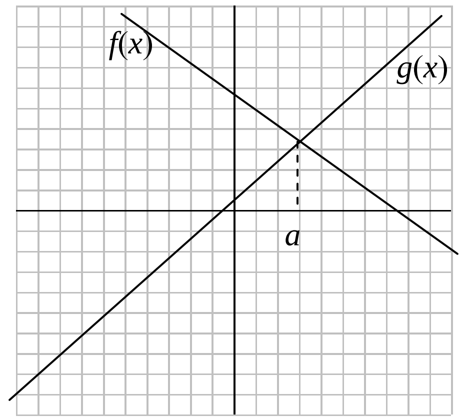


Figure 2

1) If we use the same method to solve a related equation $f(x) + 3 = g(x) + 3$, what would the graphs look like? Please sketch them in Figure 2.

2) Suppose $x = b$ is the solution to $f(x) + 3 = g(x) + 3$, what would be the relationship between a and b?

A. $a > b$ **B**. $a = b$ **C**. $a < b$ **D**. It depends on what $f(x)$ and $g(x)$ are

3) Further, if $x = c$ is the solution to another related equation, $3f(x) = 3g(x)$, what would be the relationship between a and c?

A. $a > c$ **B**. $a = c$ **C**. $a < c$ **D**. It depends on what $f(x)$ and $g(x)$ are

6.6 In which ways may students benefit from learning and using this method? Please elaborate.

Appendix 3 Interview Protocol

Thank you for being willing to participate in the interview.

We are very interested in finding out the kinds of knowledge and experiences that are useful for algebra teachers to resolve problem situations related to the teaching and learning of equation solving.

I will ask you a series of questions about solving linear equations. Most of them are tied to the paper-and-pencil assessment that you have already completed in last December. There may not be right or wrong answers, any answer and explanation that are based on your own knowledge and experiences are helpful. Please feel free to ask for clarifications whenever necessary.

Here is a consent form that explains details of our study -- the purposes, procedure, potential risks and benefits, etc. At the end you are asked to indicate whether you give permissions for this interview to be audio-taped.

1. Some general questions about linear equation solving:

- What methods have you taught for solving linear equations?

- Is there any particular method you like better than the others and want to teach to your students (i.e., they should definitely learn)?

- If so, why (what are your criteria)? Please elaborate.

- If not, why not? Are they equally important? What are the differences?

- When there is no particular method required, do you expect your students to be able to solve linear equations with any methods they like - as long as they do it correctly, or is there any one method you feel more suitable than others? Why?

- What are your ultimate goals in teaching it/them?

- In solving the same equations, have you let your students demonstrate different methods in front of the class? What new methods have you seen from your students? And further, have you let them compare the strengths of each?

Is any of the following experiences particularly helpful to you in terms of giving insights into the above issue? Please elaborate/be more specific.

1. High school math learning
2. Math preparation
3. Math education method preparation
4. Professional Development workshops
5. Collaborating with/assisting/learning from colleagues
6. Using and appraising textbooks and other instructional materials
7. Using manipulatives and technology
8. Dealing with student questions and conceptions
9. Implementing state standards / preparing for state exams

2. Specific questions about the **balancing method**:

- Do you/the textbooks explicitly state the 4 properties of equality? When and how? Is it really necessary to teach? Why or why not?

- How often do you use manipulatives/visuals (balancing scale, algebra tiles) in teaching/learning the balancing method -- what are the strengths and limitations of them?

- If going through 3 steps, how do you help students to eventually develop symbolic fluency and understanding without relying on the visual/hands-on any longer?

- In solving the equation $3x - 5 = 10$, some students subtract 5 from both sides. What would you do to help those students have better understanding?
  - → Rate different strategies, please rate their effectiveness from 1 (least) to 5 (most):

3. Specific questions about the undoing method

- Can you think of a linear equation for which the undoing method won't directly apply?
- What are the characteristics of those that can be directly solved with this method?
- What would be the benefits and challenges for student learning?
- What would be the important things for teaching?

4. Specific questions about intersecting the lines method:

- What are the advantages of the method?

- What may make it hard to learn about this method?

- What are the challenges in teaching this method?

- f = g $\leftrightarrow$ f + 3 = g + 3; Why the two solutions are (not) same?
- 3f = 3g; Why the two solutions are (not) same?
- f + 3*x* = g + 3*x*? Will the two solutions be same or not? Why (not) same? If not, which might be bigger?

5. Among the five methods, which ones are most similar to each other, in terms of

(1) the mathematics behind the process

(2) the meaning of the equal sign

Which method(s) is closer to students' notion of the equal sign in elementary school arithmetic (actions on numbers → result)?

(3) student learning and understanding

(4) types of representations used in explaining processes and giving examples.

For each criterion, group the methods by similarity criteria, and then elaborate.

Is any of the following experiences particularly helpful to you in terms of giving insights into the above issue? Please elaborate/be more specific.

1. High school math learning
2. Math preparation
3. Math education method preparation
4. Professional Development workshops
5. Collaborating with/assisting/learning from colleagues
6. Using and appraising textbooks and other instructional materials
7. Using manipulatives and technology
8. Dealing with student questions and conceptions
9. Implementing state standards / preparing for state exams

6. Rating table: rate the five methods by the 8 measures and explain through thinking aloud (***ask more questions about those that are rated 1 or 5***)

   Is any of the following experiences particularly helpful to you in terms of giving insights into the above issue? Please elaborate/be more specific.

   1. Math preparation
   2. Math method preparation
   3. Professional Development
   2. Collaborating with/assisting/learning from colleagues
   3. Using and appraising textbooks and other instructional materials
   4. Using technology
   5. Dealing with student questions and conceptions
   6. Implementing state standards / preparing for state exams

7. Is there a particular order by which you would teach the five methods? Why such order? Please elaborate.

8. We teach algorithms for the four basic operations. We teach routines / procedures / formulas for solving equations. What are the major pros and cons?
How much are students' rote memorization and lack of conceptual understanding related to the algorithms and routines themselves?

(***Be specific / ask for examples***)

*Sometimes teachers create and teach rules for equation solving (**the Golden Rule of Algebra or the balancing method, the order of operations e.g., PEMDAS, and the three steps for the undoing method**):*

<u>Be specific to PMDAS and rule for undoing:</u>

- Have you ever seen or taught this rule before? [*If the answer is no, explain the rule briefly to the interviewee*]
- Is this rule absolute? Should it be followed all the time? (e.g., can we divide first in using balancing? Do we have to combine like terms first in undoing?) give examples if not!
- In which ways/to what extent may this rule be useful for student learning? Please elaborate. How would students feel about the rule?
- In which ways/to what extent may this be rule useful for your teaching? Please elaborate.

Is any of the following experiences particularly helpful to you in terms of giving insights into the above issue? Please elaborate/be more specific.

1. Math preparation
2. Math method preparation
3. Professional Development
4. Collaborating with/assisting/learning from colleagues
5. Using and appraising textbooks and other instructional materials
6. Using technology
7. Dealing with student questions and conceptions
8. Implementing state standards / preparing for state exams

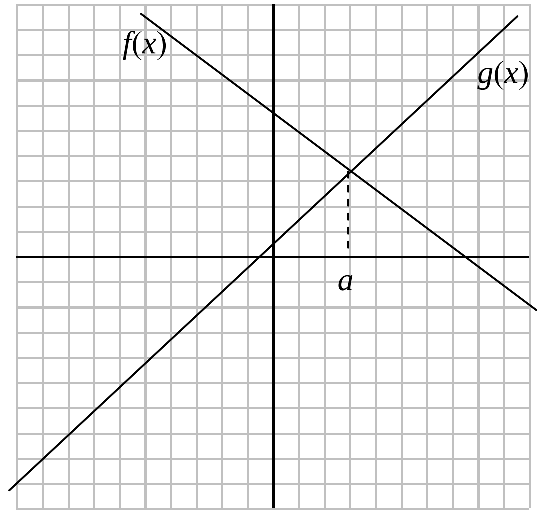
Rating Table

On a 5-point scale, please rate each of the five methods for solving linear equations in terms of the various attributes:

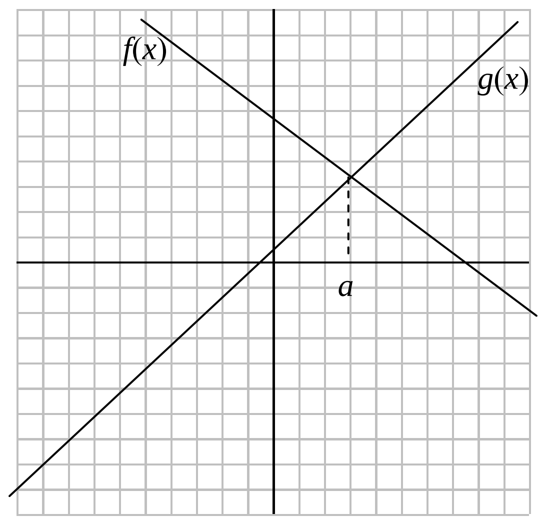
	The balancing method	The Undoing method	Tracing the line	Examining the table	Intersecting the lines
1. Accuracy 1 2 3 4 5 Least Most					
2. Generality 1 2 3 4 5 Least Most					
3. Efficiency 1 2 3 4 5 Least Most					
4. Transparency 1 2 3 4 5 Least Most					
5. Mathematical value 1 2 3 4 5 Least Most					
6. Easy to apply 1 2 3 4 5 Least Most					
7. Easy to teach 1 2 3 4 5 Least Most					
8. Easy to learn 1 2 3 4 5 Least Most					

The three most important attributes to be considered are ______________________

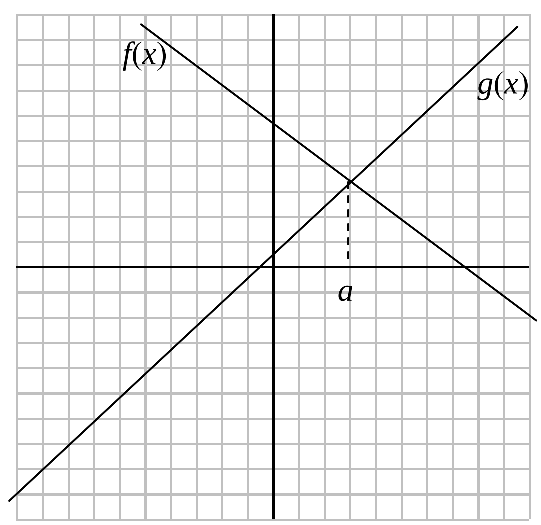
$f(x) + 3 = g(x) + 3$



$3f(x) = 3g(x)$



$f(x) + 3x = g(x) + 3x$



Which of the following experiences is/are particularly helpful to you in terms of giving insights into the above issue? Please elaborate/be more specific.

1. High school math training

2. Advanced math preparation

3. Math education method preparation

4. Professional development workshops

5. Collaborating with/assisting/learning from colleagues

6. Using and appraising algebra textbooks and other instructional materials

7. Using manipulatives and technology

8. Dealing with student questions and conceptions

9. Implementing state standards / preparing for state exams

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